A class of exactness properties characterized via left Kan extensions

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Abstract

We consider a general class of exactness properties on a finitely complete category, all of which can be expressed as the condition that a certain morphism in a diagram is a strong epimorphism. For each such exactness property, we characterize finitely bicomplete categories having the property by restricting the condition to those diagrams built from only one object in the category via a left Kan extension. In the regular context, this generalizes the theory of approximate co-operations introduced by D. Bourn and Z. Janelidze. As an application, we deduce from this a characterization of (essentially) algebraic categories satisfying such a given exactness property. The pointed version of these exactness properties is also studied.

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Introduction

In [24], we studied exactness properties on a regular category [2] which can be expressed as: 'for any diagram of a given finite shape, a given morphism between finite limits built from this diagram is a regular epimorphism'. The properties of being a Mal'tsev category [11], or more generally an n-permutable category [10] are examples of such properties. We extracted the Mal'tsev conditions characterizing varieties of universal algebras satisfying them, and generalized the theory of approximate co-operations from [8, 30, 19, 22]. The aim of the present paper is to generalize these results in several aspects.

First of all, we replace regular epimorphisms by strong epimorphisms in order to extend our theory from the regular context to the finitely complete context. Secondly, one will now be allowed to build some finite colimits from the starting diagram and some induced morphisms from them. However, we conjecture that this does not add any new examples (see Conjecture 7.6) and in each of our examples, the designated morphism required to be a strong epimorphism will be built using only finite limits and induced morphisms to them. More importantly, we allow to add new morphisms in the diagram *after* having already computed some limits or colimits. The main new example we are adding here is protomodularity (in the sense of [4]). Indeed, as shown in [6], a finitely complete category is protomodular if and only if for each diagram

$$\begin{tikzcd} & Y \arrow[d, two heads, shift right, "p"'] \\ V \arrow[r, "f"'] & W \arrow[u, shift right, "s"'] \end{tikzcd} \tag{1}$$

with $p \circ s = 1_W$, considering the pullback of f along p,

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, two heads, shift right, "p"'] \\ V \arrow[r, "f"'] & W \arrow[u, shift right, "s"'] \end{tikzcd} \tag{2}$$

the morphism f' and s are jointly strongly epimorphic, i.e., for any extension of the above diagram (2) as

$$\begin{tikzcd} & & X \arrow[dl, "q"] \\ P \arrow[r, "f'"] \arrow[d, "p'"'] \arrow[urr, bend left, "g"] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, two heads, shift right, "p"'] & \\ V \arrow[r, "f"'] & W \arrow[u, shift right, "s"'] \arrow[uur, bend right, "h"'] & \end{tikzcd}$$

with $q \circ g = f'$ and $q \circ h = s$, then the morphism q is a strong epimorphism. Notice that the morphism g here could not have been considered in the first diagram (1) since its domain is a limit built from this original diagram. In the general case, the properties we are considering will all look like: 'starting from any diagram of a given finite shape, constructing some finite (co)limits from it and induced morphisms by them, then considering any extension of a given shape of the obtained diagram, constructing some finite (co)limits and induced morphisms, then considering any extension of the obtained diagram, and so forth, then a morphism $q\colon X \to Y$ in the resulting diagram should be a strong epimorphism'. Using the formalism of exactness sequents introduced in [23], we can represent such an exactness property by a sequence of subsketch inclusions

$$\begin{tikzcd} \varnothing \arrow[r, "\alpha_1"] \arrow[rrrrrr, bend right, "\alpha"'] & \mathcal{A}_1 \arrow[r, "\beta_1"] & \mathcal{B}_1 \arrow[r, "\alpha_2"] & \cdots \arrow[r, "\alpha_n"] & \mathcal{A}_n \arrow[r, "\beta_n"] & \mathcal{B}_n \arrow[r, "\omega"] & \mathcal{A} \arrow[r, "\beta"] & \mathcal{B} \end{tikzcd}$$

where $\mathcal{A}_1$ represents the shape of the initial diagram and each $\mathcal{A}_i$ $(2 \leqslant i \leqslant n)$ represents a further extension. The sketches $\mathcal{B}_i$ encode the (co)limits and the induced morphisms to consider. Of course some restrictions apply to this; they are listed by conditions (Ax 0)–(Ax 5) in Section 1. Among them, (Ax 5) requires that the codomain Y of $q \in \mathcal{B}_n$ represents a limit of a finite diagram $D_Y\colon \mathcal{H}_Y \to \mathcal{A}_1$ in the initial sketch $\mathcal{A}_1$.

Given such an exactness property, we explain how to construct, from any object C in a finitely bicomplete category $\mathbb{C}$, a universal $\mathcal{B}_n$-structure L_C by means of successive pointwise left Kan extensions. This $\mathcal{B}_n$-structure is the reflection of the constant diagram $\Delta_C\colon \mathcal{H}_Y \to \mathbb{C}$ along the composition functor

$$(D_Y^{\mathcal{B}_n})_{\mathbb{C}}\colon \mathcal{B}_n\mathbb{C} \to \mathcal{H}_Y\mathbb{C}$$

where $D_Y^{\mathcal{B}_n} = \beta_n \circ \alpha_n \circ \cdots \circ \beta_1 \circ D_Y$ and is thus denoted by $\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_C$ in the text. For the protomodularity example, this $\mathcal{B}_n$-structure L_C is given by

$$\begin{tikzcd}
 & & (0 \times_{!_C,\nabla_C} 2C) + C \arrow[dl, "\binom{p_2}{\iota_2}"'] \\
0 \times_{!_C,\nabla_C} 2C \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[urr, bend left, "\iota_1'"] \arrow[dr, phantom, "\lrcorner", very near start] & 2C \arrow[d, shift right, "\nabla_C"'] & \\
0 \arrow[r, "!_C"'] & C \arrow[u, shift right, "\iota_2"'] \arrow[uur, bend right, "\iota_2'"'] &
\end{tikzcd}$$

where $0 \times_{!_C,\nabla_C} 2C$ is the pullback of the codiagonal $\nabla_C \colon 2C \to C$ along the unique morphism $0 \to C$ from the initial object. Our main result, Theorem 3.1, states that $\mathbb{C}$ satisfies the exactness property if and only if, for each object C, the morphism $L_C(q)$ is a strong epimorphism. Moreover, in the regular finitely cocomplete context, this is equivalent to require that, for each object C, the pullback γ_C of $L_C(q)$ along the universal morphism $e_C \colon C \to L_C(Y)$ is a regular epimorphism.

$$\begin{tikzcd}
\mathrm{Ap}(C) \arrow[r, "\delta_C"] \arrow[d, "\gamma_C"'] \arrow[dr, phantom, "\lrcorner", very near start] & L_C(X) \arrow[d, "L_C(q)"] \\
C \arrow[r, "e_C"'] & L_C(Y)
\end{tikzcd}$$

This result generalizes the characterization via approximate co-operations from [9] in the protomodular case and from [8, 30, 19, 22] in the case of matrix conditions, where the morphism $\gamma_C \colon \mathrm{Ap}(C) \to C$ is called an 'approximation'. Using an appropriate presentation of the Mal'tsev property, we also deduce from this theorem that a finitely bicomplete category is a Mal'tsev category [12] if and only if for any object C, the morphism $\begin{pmatrix} \iota_1 & \iota_2 \\ \iota_2 & \iota_2 \\ \iota_2 & \iota_1 \\ \iota_2 & \iota_2 \end{pmatrix} \colon 4C \to \mathrm{Eq}[\nabla_C]$ is a strong epimorphism, where $\mathrm{Eq}[\nabla_C]$ is the kernel pair of the codiagonal ∇_C.

The validity of such an exactness property in the particular case of a variety of universal algebras can be further reduced to the condition that, considering the free algebra $\mathrm{Fr}(\{x\})$ on one variable x, the variable x is in the image of $\gamma_{\mathrm{Fr}(\{x\})}$, i.e., that $e_{\mathrm{Fr}(\{x\})}(x)$ is in the image of $L_{\mathrm{Fr}(\{x\})}(q)$ (see Theorem 4.1). It seems we can easily extract from this result a Mal'tsev condition characterizing varieties with the given property, generalizing many such characterizations, e.g., those from [7, 16, 32, 34]. However, we have not yet been able to formally prove this always gives rise to a Mal'tsev condition (see Conjecture 7.5).

Finally, we also consider the pointed version of this class of exactness properties and mention the counterpart of the above theorems in the pointed context. As additional examples, one gets the following properties (some of which requiring to be considered in the regular finitely cocomplete pointed context): being a normal category [31], having normal projections [26], having products commuting with coequalizers, and all the pointed matrix conditions [29] (including the examples of being unital [5], strongly unital [5] and subtractive [27]).

The paper is organized as follows. We formally describe in Section 1 the exactness properties we are studying in this paper. To this end, we need to recall some aspects of the theory of exactness sequents from [23]. In Section 2, for each such exactness property, we build the universal $\mathcal{B}_n$-structure associated to each object of a finitely bicomplete category. Section 3 is devoted to the characterization of finitely bicomplete categories satisfying such an exactness property by means of this universal $\mathcal{B}_n$-structure associated to each object.

This also gives rise to the generalization of the theory of approximate co-operations in the regular context. In Section 4, we further extend this characterization in the varietal case, giving an easy way to find the algebraic condition associated to each of these exactness properties. Section 5 is devoted to the study of some examples: protomodularity, involution-rigidness [32] and the Mal'tsev property. We then treat the pointed context in Section 6 and give additional examples in that context. In Section 7, we make further remarks, among which we give a generalization of the characterization in the varietal case to the case of essentially algebraic categories [1]. We end Section 7 with some conjectures.

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1 Some exactness properties

In this section, we formally describe the exactness properties we are going to work with. In order to do so, we need to recall some notions from [23, 24] (the interested reader is referred to those references for more details on these notions). For a graph $\mathcal{G}$ (i.e., a diagram $d, c\colon E \rightrightarrows V$ in the category **Set** of sets), a *commutativity condition* in $\mathcal{G}$ is a pair of paths

$$((A_0, f_1, A_1, \dots, f_m, A_m), (B_0, g_1, B_1, \dots, g_{m'}, B_{m'}))$$

in $\mathcal{G}$ such that $A_0 = B_0$ and $A_m = B_{m'}$. We represent it by

$$f_m \cdots f_1 = g_{m'} \cdots g_1$$

or by

$$f_m \cdots f_1 = 1_{B_0}$$

if $m' = 0$ (and similarly if $m = 0$). A *finite diagram* in $\mathcal{G}$ is given by a finite graph $\mathcal{H}$ together with a morphism of graphs $D\colon \mathcal{H} \to \mathcal{G}$. A *finite limit condition* (respectively, a *finite colimit condition*) in $\mathcal{G}$ is an equivalence class of 4-tuples $(\mathcal{H}, D, C, (c_H)_{H\in\mathcal{H}})$ where $D\colon \mathcal{H} \to \mathcal{G}$ is a finite diagram, C is an object in $\mathcal{G}$ and for each object H in $\mathcal{H}$, $c_H\colon C \to D(H)$ (respectively, $c_H\colon D(H) \to C$) is an arrow in $\mathcal{G}$. Two such 4-tuples $(\mathcal{H}, D, C, (c_H)_{H\in\mathcal{H}})$ and $(\mathcal{H}', D', C', (c'_{H'})_{H'\in\mathcal{H}'})$ are considered to be equivalent if $C = C'$ and if there exists an isomorphism of graphs $I\colon \mathcal{H} \to \mathcal{H}'$ such that $D'I = D$ and $c_H = c'_{I(H)}$ for any $H \in \mathcal{H}$. Such a condition $[(\mathcal{H}, D, C, (c_H)_{H\in\mathcal{H}})]$ is represented by

$$(C, (c_H)_H) = \mathbf{limit}(\mathcal{H}, D) \quad \text{(respectively by } (C, (c_H)_H) = \mathbf{colimit}(\mathcal{H}, D)).$$

Finite limit conditions and finite colimit conditions are called *convergence conditions.* A *sketch* is then a finite graph equipped with a set of commutativity conditions and a set of convergence conditions. A *morphism of sketches* is a morphism $\mu\colon \mathcal{G} \to \mathcal{G}'$ of underlying graphs of sketches which carries each commutativity condition on $\mathcal{G}$ to a commutativity condition on $\mathcal{G}'$ and each convergence condition on $\mathcal{G}$ to a convergence condition on $\mathcal{G}'$. We denote by

$$\mathscr{G}\colon \mathbf{Sk} \to \mathbf{FGraph}$$

the forgetful functor from the category of sketches to the category of finite graphs. A *subsketch* of a sketch $\mathcal{B}$ is a subgraph $\mathcal{A}$ of the underlying graph of $\mathcal{B}$ equipped with a sketch structure that turns the inclusion of graphs $\mathcal{A} \to \mathcal{B}$ into a sketch morphism. Such morphisms are called *subsketch inclusions*.

Given a sketch $\mathcal{A}$ and a category $\mathbb{C}$, an $\mathcal{A}$*-structure* in $\mathbb{C}$ is a morphism $F\colon \mathcal{A} \to \mathbb{C}$ of underlying graphs which carries each commutativity condition of $\mathcal{A}$ to an actual commutative diagram in $\mathbb{C}$, and each finite limit/colimit condition of $\mathcal{A}$ to an actual limit/colimit in $\mathbb{C}$. The category of $\mathcal{A}$-structures in $\mathbb{C}$ (and natural transformations as morphisms) is denoted by $\mathcal{A}\mathbb{C}$. Every morphism $\beta\colon \mathcal{A} \to \mathcal{B}$ of sketches gives rise to a functor

$$\beta_{\mathbb{C}}\colon \mathcal{B}\mathbb{C} \to \mathcal{A}\mathbb{C}$$

by 'pre-composing' with β.

An *exactness sequent* (called an $\varnothing$-sequent in [23]) is a subsketch inclusion $\beta\colon \mathcal{A} \to \mathcal{B}$ considered as a sequence of subsketch inclusions

$$\varnothing \xrightarrow{\alpha} \mathcal{A} \xrightarrow{\beta} \mathcal{B}$$

starting with the empty sketch $\varnothing$. We denote such a sequent by $\alpha \vdash \beta$. As detailed below, we are only interested in particular exactness sequents where β is 'constructible'. In that case, $\beta_{\mathbb{C}}$ is fully faithful for any category $\mathbb{C}$, and we write '$\alpha \vdash_{\mathbb{C}} \beta$' when $\beta_{\mathbb{C}}$ is an equivalence of categories (see Lemma 1.4 in [23]).

Each finite category $\mathbb{A}$ induces a sketch $U(\mathbb{A})$ whose underlying graph is the underlying graph of $\mathbb{A}$ and whose commutativity conditions are:

- $((A, f, B, g, C), (A, g\circ f, C))$ for any pair of composable arrows $f\colon A \to B$ and $g\colon B \to C$ in $\mathbb{A}$;
- $((A, 1_A, A), (A))$ for any object A in $\mathbb{A}$.

There are no convergence conditions on $U(\mathbb{A})$. A $U(\mathbb{A})$-structure in a category $\mathbb{C}$ thus corresponds to a functor $\mathbb{A} \to \mathbb{C}$. A subsketch inclusion $\alpha\colon \mathcal{B} \to \mathcal{A}$ is said to be *unconditional of finite kind* when $\mathcal{A}$ is the underlying sketch $U(\mathbb{A})$ of a finite category $\mathbb{A}$ further equipped with the commutativity and convergence conditions which already appear in $\mathcal{B}$.

If $\mathcal{A}$ is a sketch with underlying graph $\mathcal{G}$ and if $f_m \cdots f_1 = g_{m'} \cdots g_1$ is a commutativity condition in the graph $\mathcal{G}$ (one that is not necessarily included in $\mathcal{A}$), we write

$$f_m \cdots f_1 \equiv_{\mathcal{A}} g_{m'} \cdots g_1$$

if, for any $\mathcal{A}$-structure F in any category $\mathbb{C}$, the equality

$$F(f_m) \circ \cdots \circ F(f_1) = F(g_{m'}) \circ \cdots \circ F(g_1)$$

holds. Similarly, if $D\colon \mathcal{H} \to \mathcal{G}$ is a finite diagram in $\mathcal{G}$ and if $(p_H\colon C \to D(H))_{H\in\mathcal{H}}$ is a family of paths in $\mathcal{G}$ indexed by the objects of $\mathcal{H}$, we write

$$(C, (p_H)_H) \equiv_{\mathcal{A}} \mathbf{limit}(\mathcal{H}, D)$$

if, for any $\mathcal{A}$-structure F in any category $\mathbb{C}$, the cone $(F(C), (F(p_H))_H)$ is an actual limit of $F \circ D$ in $\mathbb{C}$. As usual, $F(p_H)$ represents the composite in $\mathbb{C}$ of the actual images under F of the arrows constituting the path p_H. We also use an analogous notation for colimits.

The exactness sequents $\alpha \vdash \beta$ we are going to consider in this paper can be decomposed as sequences of subsketch inclusions as in

$$\begin{tikzcd} \varnothing \arrow[r, "\alpha_1"] \arrow[rrrrrrr, bend right, "\alpha"'] & \mathcal{A}_1 \arrow[r, "\beta_1"] & \mathcal{B}_1 \arrow[r, "\alpha_2"] & \cdots \arrow[r, "\alpha_n"] & \mathcal{A}_n \arrow[r, "\beta_n"] & \mathcal{B}_n \arrow[r, "\omega"] & \mathcal{A} \arrow[r, "\beta"] & \mathcal{B} \end{tikzcd} \tag{3}$$

where

(Ax 0) $n \geqslant 1$ is a natural number, $\alpha = \omega \circ \beta_n \circ \alpha_n \circ \cdots \circ \beta_1 \circ \alpha_1$ and $\mathcal{B}_0 = \varnothing$ is the empty sketch;

(Ax 1) for each $i \in \{1, \ldots, n\}$, the subsketch inclusion $\alpha_i \colon \mathcal{B}_{i-1} \to \mathcal{A}_i$ is unconditional of finite kind (i.e., $\mathcal{A}_i$ is the underlying sketch of a finite category $\mathbb{A}_i$ further equipped with the commutativity and convergence conditions coming from $\mathcal{B}_{i-1}$);

(Ax 2) for each $i \in \{2, \ldots, n\}$ and for each morphism $f \colon B \to B'$ in $\mathbb{A}_i$ such that $B, B' \in \mathcal{B}_{i-1}$, there exists a path $p \colon B \to B'$ in $\mathcal{B}_{i-1}$ such that the composite of p in $\mathbb{A}_i$ is f and if $p, p' \colon B \to B'$ are two such paths then $p \equiv_{\mathcal{B}_{i-1}} p'$;

(Ax 3) for each $i \in \{1, \ldots, n\}$, the subsketch inclusion $\beta_i \colon \mathcal{A}_i \to \mathcal{B}_i$ can be decomposed as a finite sequence

$$\mathcal{A}_i = \mathcal{B}_{i,0} \longrightarrow \mathcal{B}_{i,1} \longrightarrow \mathcal{B}_{i,2} \longrightarrow \cdots \longrightarrow \mathcal{B}_{i,k_i} = \mathcal{B}_i$$

of subsketch inclusions (with $k_i \geqslant 0$), where every next subsketch $\mathcal{B}_{i,j+1}$ of $\mathcal{B}_i$ is obtained from the previous subsketch $\mathcal{B}_{i,j}$ by any one of the following procedures:

(a) include some commutativity conditions $f_m \cdots f_1 = g_{m'} \cdots g_1$ expressed using objects and arrows which belong to $\mathcal{B}_{i,j}$ and for which $f_m \cdots f_1 \equiv_{\mathcal{B}_{i,j}} g_{m'} \cdots g_1$ holds (in other words, include some redundant commutativity conditions),

(b) include some convergence conditions $(C, (c_H)_H) = \mathbf{limit}(\mathcal{H}, D)$ (or $(C, (c_H)_H) = \mathbf{colimit}(\mathcal{H}, D)$ respectively) where C, the c_H's and D all belong to $\mathcal{B}_{i,j}$ and for which $(C, (c_H)_H) \equiv_{\mathcal{B}_{i,j}} \mathbf{limit}(\mathcal{H}, D)$ holds (respectively $(C, (c_H)_H) \equiv_{\mathcal{B}_{i,j}} \mathbf{colimit}(\mathcal{H}, D)$) (in other words, include some redundant convergence conditions),

(c) include an arrow f, not already in $\mathcal{B}_{i,j}$, and a commutativity condition $f = g_m \cdots g_1$ where $m \geqslant 0$ and where $g_1, \ldots, g_m$ are arrows in $\mathcal{B}_{i,j}$,

(d) include an object C, not already in $\mathcal{B}_{i,j}$, together with the pairwise distinct arrows c_H's and the condition $(C, (c_H)_H) = \mathbf{limit}(\mathcal{H}, D)$, where D is a finite diagram in $\mathcal{B}_{i,j}$,

(e) given in $\mathcal{B}_{i,j}$ a condition $(C, (c_H)_H) = \mathbf{limit}(\mathcal{H}, D)$, an object A, a family $(a_H \colon A \to D(H))_{H \in \mathcal{H}}$ of arrows and commutativity conditions $D(h) \cdot a_H = a_{H'}$ for each arrow $h \colon H \to H'$ in $\mathcal{H}$, include an arrow $f \colon A \to C$, not already in $\mathcal{B}_{i,j}$, and commutativity conditions $c_H \cdot f = a_H$ for each object $H \in \mathcal{H}$,

(f) include an object C, not already in $\mathcal{B}_{i,j}$, together with the pairwise distinct arrows c_H's and the condition $(C, (c_H)_H) = \mathbf{colimit}(\mathcal{H}, D)$, where D is a finite diagram in $\mathcal{B}_{i,j}$,

(g) given in $\mathcal{B}_{i,j}$ a condition $(C, (c_H)_H) = \mathbf{colimit}(\mathcal{H}, D)$, an object A, a family $(a_H \colon D(H) \to A)_{H \in \mathcal{H}}$ of arrows and commutativity conditions $a_{H'} \cdot D(h) = a_H$ for each arrow $h \colon H \to H'$ in $\mathcal{H}$, include an arrow $f \colon C \to A$, not already in $\mathcal{B}_{i,j}$, and commutativity conditions $f \cdot c_H = a_H$ for each object $H \in \mathcal{H}$;

(Ax 4) there exists in $\mathcal{B}_n$ an arrow $q\colon X \to Y$ such that

1. the subsketch inclusion $\omega\colon \mathcal{B}_n \to \mathcal{A}$ is constructed by adding to $\mathcal{B}_n$ an object Y' (not already in $\mathcal{B}_n$) and arrows $l_q\colon X \to Y'$, $m_q\colon Y' \to Y$ and $i_{Y'}\colon Y' \to Y'$, together with the commutativity conditions $q = m_q \cdot l_q$ and $i_{Y'} = 1_{Y'}$ and the convergence condition $(Y', (i_{Y'}, i_{Y'}, m_q)) = \mathbf{limit}(\mathcal{H}_{\mathrm{Pb}}, D_{\mathrm{Pb}}^{m_q,m_q})$ where $\mathcal{H}_{\mathrm{Pb}}$ is the graph
$$\begin{tikzcd} & V_2 \arrow[d, "v_2"] \\ V_1 \arrow[r, "v_1"'] & V_3 \end{tikzcd}$$
and $D_{\mathrm{Pb}}^{m_q,m_q}\colon \mathcal{H}_{\mathrm{Pb}} \to \mathcal{A}$ is defined via $D_{\mathrm{Pb}}^{m_q,m_q}(v_1) = m_q = D_{\mathrm{Pb}}^{m_q,m_q}(v_2)$,
2. the subsketch inclusion $\beta\colon \mathcal{A} \to \mathcal{B}$ is obtained by adding to $\mathcal{A}$ the convergence condition $(Y', (m_q)) = \mathbf{limit}(\mathcal{H}_*, D_*^Y)$ where $\mathcal{H}_*$ is the graph with a single object $*$ and no arrows and D_*^Y is defined by $D_*^Y(*) = Y$;

(Ax 5) for the codomain Y of q, there exists a finite diagram $D_Y\colon \mathcal{H}_Y \to \mathcal{A}_1$ in $\mathcal{A}_1$ and a family of paths $(p_H^Y\colon Y \to D_Y^{\mathcal{B}_n}(H))_{H\in\mathcal{H}_Y}$ in $\mathcal{B}_n$ such that

$$(Y, (p_H^Y)_H) \equiv_{\mathcal{B}_n} \mathbf{limit}(\mathcal{H}_Y, D_Y^{\mathcal{B}_n})$$

where we denoted $\beta_n \circ \alpha_n \circ \cdots \circ \beta_1 \circ D_Y$ by $D_Y^{\mathcal{B}_n}$ for the sake of brevity.

By Lemma 1.2 in [24], the condition (Ax 5) is always satisfied if the following two further conditions are satisfied:

- $Y \in \mathcal{B}_1$;
- in the decomposition of $\beta_1\colon \mathcal{A}_1 \to \mathcal{B}_1$ as in condition (Ax 3), procedures (b), (f) and (g) are not used.

The conditions (Ax 3) and (Ax 4) imply in particular that the subsketch inclusions β_i (for each $i \in \{1, \ldots, n\}$) and β are *constructible* in the sense of [23]. This implies that, for any category $\mathbb{C}$, the functors $(\beta_1)_{\mathbb{C}}, \ldots, (\beta_n)_{\mathbb{C}}$ and $\beta_{\mathbb{C}}$ are fully faithful. Moreover, if the category $\mathbb{C}$ is finitely complete and finitely cocomplete, the functors $(\beta_1)_{\mathbb{C}},\ldots,(\beta_n)_{\mathbb{C}}$ are equivalences of categories. We denote by $(\beta_i)_{\mathbb{C}}^{-1}\colon \mathcal{A}_i\mathbb{C} \to \mathcal{B}_i\mathbb{C}$ their unique (up to isomorphism) pseudo-inverses, which are chosen such that $(\beta_i)_{\mathbb{C}} \circ (\beta_i)_{\mathbb{C}}^{-1} = 1_{\mathcal{A}_i\mathbb{C}}$ (but we only have an isomorphism $(\beta_i)_{\mathbb{C}}^{-1} \circ (\beta_i)_{\mathbb{C}} \cong 1_{\mathcal{B}_i\mathbb{C}}$). For an $\mathcal{A}_i$-structure F_i in $\mathbb{C}$, we denote by $\overline{F_i}$ the $\mathcal{B}_i$-structure $(\beta_i)_{\mathbb{C}}^{-1}(F_i)$, which is obtained from F_i by constructing finite limits, finite colimits, induced morphisms to those limits or from those colimits and composite morphisms as prescribed by the step by step construction of β_i described in (Ax 3). By abuse of notation, we will also denote by F_i the functor $F_i\colon \mathbb{A}_i \to \mathbb{C}$ corresponding to the $\mathcal{A}_i$-structure F_i.

The condition $\alpha \vdash_{\mathbb{C}} \beta$ for a category $\mathbb{C}$ can be stated as: 'for any $\mathcal{A}$-structure F in $\mathbb{C}$, the morphism $F(m_q)$ is an isomorphism'. In a category with finite limits, an *extremal epimorphism* (or equivalently a *strong epimorphism*) is a morphism p such that for any factorization $p = m \circ l$ with m a monomorphism, one has that m is an isomorphism. Therefore, if the category $\mathbb{C}$ is finitely complete, the condition $\alpha \vdash_{\mathbb{C}} \beta$ can be equivalently rephrased as: 'for any $\mathcal{B}_n$-structure $\overline{F_n}$ in $\mathbb{C}$, the morphism $\overline{F_n}(q)$ is a strong epimorphism'. If $\mathbb{C}$ is finitely complete and finitely cocomplete, it can even be formulated as: 'for any functor $F_1\colon \mathbb{A}_1 \to \mathbb{C}$, considering its extension $\overline{F_1}$ as a $\mathcal{B}_1$-structure, for any extension of it as a functor $F_2\colon \mathbb{A}_2 \to \mathbb{C}$, considering its extension $\overline{F_2}$ as a $\mathcal{B}_2$-structure, ..., for any extension of $\overline{F_{n-1}}$ as a functor $F_n\colon \mathbb{A}_n \to \mathbb{C}$, considering its extension $\overline{F_n}$ as a $\mathcal{B}_n$-structure, then $\overline{F_n}(q)$ is a strong epimorphism'.

$$\begin{tikzcd} \varnothing \arrow[r, "\alpha_1"] & \mathcal{A}_1 \arrow[r, "\beta_1"] \arrow[drr, "\forall F_1"'] & \mathcal{B}_1 \arrow[r, "\alpha_2"] \arrow[dr, dotted, "\overline{F_1}"] & \cdots \arrow[r, "\alpha_n"] & \mathcal{A}_n \arrow[r, "\beta_n"] \arrow[dl, "\forall F_n"'] & \mathcal{B}_n \arrow[r, "\omega"] \arrow[dll, dotted, "\overline{F_n}"] & \mathcal{A} \arrow[r, "\beta"] & \mathcal{B} \\ & & & \mathbb{C} & & & & \end{tikzcd}$$

2 Kan extensions

Let us suppose we are given a finite graph $\mathcal{H}$ and a diagram $D\colon \mathcal{H} \to \mathbb{A}$ in a finite category $\mathbb{A}$. By abuse of notation, we also denote by $\mathcal{H}$ the sketch whose underlying graph is $\mathcal{H}$ and which has no conditions and by $D\colon \mathcal{H} \to U(\mathbb{A})$ the sketch morphism induced by the diagram D. Given a finitely cocomplete category $\mathbb{C}$, the functor $D_{\mathbb{C}}\colon U(\mathbb{A})\mathbb{C} \to \mathcal{H}\mathbb{C}$ has a left adjoint $\operatorname{Lan}_D\colon \mathcal{H}\mathbb{C} \to U(\mathbb{A})\mathbb{C}$, which can be computed via the pointwise left Kan extension formula. We recall that given an $\mathcal{H}$-structure E in $\mathbb{C}$ (i.e., a diagram $E\colon \mathcal{H} \to \mathbb{C}$), the left Kan extension $\operatorname{Lan}_D E$ is constructed as follows. For an object $A \in \mathbb{A}$, we consider the finite graph $(D \downarrow A)$ whose objects are pairs (H, f) with $H \in \mathcal{H}$ and $f\colon D(H) \to A \in \mathbb{A}$ and whose arrows $h\colon (H, f) \to (H', f')$ are arrows $h\colon H \to H'$ in $\mathcal{H}$ such that $f' \circ D(h) = f$. The object $(\operatorname{Lan}_D E)(A)$ is the colimit of the diagram $E \circ \varphi_A\colon (D \downarrow A) \to \mathbb{C}$ where $\varphi_A\colon (D \downarrow A) \to \mathcal{H}$ is the forgetful morphism of graphs. If we denote by $s^A_{(H,f)}\colon E(H) \to (\operatorname{Lan}_D E)(A)$ the legs of the colimits, the image of a morphism $a\colon A \to A'$ in $\mathbb{A}$ under $\operatorname{Lan}_D E$ is uniquely determined by $(\operatorname{Lan}_D E)(a) \circ s^A_{(H,f)} = s^{A'}_{(H,a\circ f)}$ for each $(H, f) \in (D \downarrow A)$. The E-component $\lambda^E\colon E \to (\operatorname{Lan}_D E) \circ D$ of the unit is determined by $\lambda^E_H = s^{D(H)}_{(H,1_{D(H)})}$ for each object $H \in \mathcal{H}$.

$$\begin{tikzcd} \mathcal{H} \arrow[r, "D"] \arrow[dr, "E"'] & U(\mathbb{A}) \arrow[d, "\operatorname{Lan}_D E"] \arrow[dl, phantom, "\Uparrow \lambda^E"] \\ & \mathbb{C} \end{tikzcd}$$

Lemma 2.1. *Let $\alpha\colon \mathcal{B} \to \mathcal{A}$ be a subsketch inclusion satisfying*

(i) α is unconditional of finite kind (i.e., $\mathcal{A}$ is the underlying sketch of a finite category $\mathbb{A}$ further equipped with the commutativity and convergence conditions coming from $\mathcal{B}$);

(ii) for each morphism $f\colon B \to B'$ in $\mathbb{A}$ such that $B, B' \in \mathcal{B}$, there exists a path $p\colon B \to B'$ in $\mathcal{B}$ such that the composite of p in $\mathbb{A}$ is f and if $p, p'\colon B \to B'$ are two such paths then $p \equiv_{\mathcal{B}} p'$.

Let $\mathbb{C}$ be a finitely cocomplete category. Then the functor $\alpha_{\mathbb{C}}\colon \mathcal{A}\mathbb{C} \to \mathcal{B}\mathbb{C}$ has a left-adjoint-right-inverse $\operatorname{Lan}_\alpha\colon \mathcal{B}\mathbb{C} \to \mathcal{A}\mathbb{C}$.

Proof. We consider the following commutative diagram of subsketch inclusions

$$\begin{tikzcd} \mathscr{G}(\mathcal{B}) \arrow[r, "\alpha'"] \arrow[d, "i^{\mathcal{B}}"'] & U(\mathbb{A}) \arrow[d, "j^{\mathcal{A}}"] \\ \mathcal{B} \arrow[r, "\alpha"'] & \mathcal{A} \end{tikzcd}$$

where the underlying graph $\mathscr{G}(\mathcal{B})$ of $\mathcal{B}$ is considered as a sketch with no conditions. As explained above, the functor $\alpha'_{\mathbb{C}}\colon U(\mathbb{A})\mathbb{C} \to \mathscr{G}(\mathcal{B})\mathbb{C}$ has a left adjoint $\mathrm{Lan}_{\alpha'}\colon \mathscr{G}(\mathcal{B})\mathbb{C} \to U(\mathbb{A})\mathbb{C}$. We denote the unit of this adjunction by $\eta\colon 1_{\mathscr{G}(\mathcal{B})\mathbb{C}} \to \alpha'_{\mathbb{C}} \circ \mathrm{Lan}_{\alpha'}$ and the counit by $\varepsilon\colon \mathrm{Lan}_{\alpha'} \circ \alpha'_{\mathbb{C}} \to 1_{U(\mathbb{A})\mathbb{C}}$. Given $E \in \mathcal{B}\mathbb{C}$ and $B' \in \mathcal{B}$, we can consider, for each $(B, f) \in (\alpha' \downarrow \alpha'(B'))$, the morphism $s^{\alpha'(B')}_{(B,f)} = E(p)\colon E(B) \to E(B')$ in $\mathbb{C}$ where $p\colon B \to B'$ is any path in $\mathcal{B}$ whose composite in $\mathbb{A}$ is f as given by our assumption (ii). In view of the uniqueness part of the assumption (ii), these morphisms are well defined and, using that $s^{\alpha'(B')}_{(B',1_{\alpha'(B')})} = 1_{E(B')}$, it is routine to show that they form a colimit cocone under $i^{\mathcal{B}}_{\mathbb{C}}(E) \circ \varphi_{\alpha'(B')}$. Since α' is injective on objects, we can thus choose $(\mathrm{Lan}_{\alpha'}\, i^{\mathcal{B}}_{\mathbb{C}}(E))(\alpha'(B'))$ to be $E(B')$. Therefore, for each $E \in \mathcal{B}\mathbb{C}$, we can choose $\eta_{i^{\mathcal{B}}_{\mathbb{C}}(E)}$ to be the identity on $i^{\mathcal{B}}_{\mathbb{C}}(E)$. We thus suppose without loss of generality that $\eta \bullet i^{\mathcal{B}}_{\mathbb{C}} = 1_{i^{\mathcal{B}}_{\mathbb{C}}}$ and $\alpha'_{\mathbb{C}} \circ \mathrm{Lan}_{\alpha'} \circ i^{\mathcal{B}}_{\mathbb{C}} = i^{\mathcal{B}}_{\mathbb{C}}$. By assumption (i), we know that the right bottom rectangle in the diagram of functors below is a pullback.

$$\begin{tikzcd}
\mathcal{B}\mathbb{C} \arrow[dr, dotted, "\mathrm{Lan}_\alpha"] \arrow[drr, bend left, "1_{\mathcal{B}\mathbb{C}}"] \arrow[ddr, bend right, "\mathrm{Lan}_{\alpha'} \circ i^{\mathcal{B}}_{\mathbb{C}}"'] & & \\
& \mathcal{A}\mathbb{C} \arrow[r, "\alpha_{\mathbb{C}}"] \arrow[d, "j^{\mathcal{A}}_{\mathbb{C}}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B}\mathbb{C} \arrow[d, "i^{\mathcal{B}}_{\mathbb{C}}"] \\
& U(\mathbb{A})\mathbb{C} \arrow[r, "\alpha'_{\mathbb{C}}"'] & \mathscr{G}(\mathcal{B})\mathbb{C}
\end{tikzcd}$$

Since the outer part of the diagram commutes, there exists a unique dotted functor making the diagram commutative. We consider the natural transformation

$$\varepsilon \bullet j^{\mathcal{A}}_{\mathbb{C}}\colon\ \ j^{\mathcal{A}}_{\mathbb{C}} \circ \mathrm{Lan}_\alpha \circ \alpha_{\mathbb{C}} = \mathrm{Lan}_{\alpha'} \circ \alpha'_{\mathbb{C}} \circ j^{\mathcal{A}}_{\mathbb{C}} \longrightarrow j^{\mathcal{A}}_{\mathbb{C}}$$

satisfying

$$\alpha'_{\mathbb{C}} \bullet \varepsilon \bullet j^{\mathcal{A}}_{\mathbb{C}} = \big(\alpha'_{\mathbb{C}} \bullet \varepsilon \bullet j^{\mathcal{A}}_{\mathbb{C}}\big) \circ \big(\eta \bullet \big(i^{\mathcal{B}}_{\mathbb{C}} \circ \alpha_{\mathbb{C}}\big)\big) = \big(\alpha'_{\mathbb{C}} \bullet \varepsilon \bullet j^{\mathcal{A}}_{\mathbb{C}}\big) \circ \big(\eta \bullet \big(\alpha'_{\mathbb{C}} \circ j^{\mathcal{A}}_{\mathbb{C}}\big)\big) = 1_{\alpha'_{\mathbb{C}} \circ j^{\mathcal{A}}_{\mathbb{C}}}.$$

Since pullbacks of functors are 'strong' (see e.g. [23]), there exists a unique natural transformation $\lambda\colon \mathrm{Lan}_\alpha \circ \alpha_{\mathbb{C}} \to 1_{\mathcal{A}\mathbb{C}}$ such that $j^{\mathcal{A}}_{\mathbb{C}} \bullet \lambda = \varepsilon \bullet j^{\mathcal{A}}_{\mathbb{C}}$ and $\alpha_{\mathbb{C}} \bullet \lambda = 1_{\alpha_{\mathbb{C}}}$. To prove that Lan_α is the left adjoint of $\alpha_{\mathbb{C}}$ with unit $1_{1_{\mathcal{B}\mathbb{C}}}$ and counit λ, it remains to prove that $\lambda \bullet \mathrm{Lan}_\alpha = 1_{\mathrm{Lan}_\alpha}$. Again since pullbacks of functors are strong, it suffices to notice that

$$\begin{aligned}
j^{\mathcal{A}}_{\mathbb{C}} \bullet \lambda \bullet \mathrm{Lan}_\alpha &= \varepsilon \bullet \big(j^{\mathcal{A}}_{\mathbb{C}} \circ \mathrm{Lan}_\alpha\big)\\
&= \varepsilon \bullet \big(\mathrm{Lan}_{\alpha'} \circ i^{\mathcal{B}}_{\mathbb{C}}\big)\\
&= \big(\varepsilon \bullet \big(\mathrm{Lan}_{\alpha'} \circ i^{\mathcal{B}}_{\mathbb{C}}\big)\big) \circ \big(\mathrm{Lan}_{\alpha'} \bullet \eta \bullet i^{\mathcal{B}}_{\mathbb{C}}\big)\\
&= 1_{\mathrm{Lan}_{\alpha'} \circ i^{\mathcal{B}}_{\mathbb{C}}}
\end{aligned}$$

and $\alpha_{\mathbb{C}} \bullet \lambda \bullet \mathrm{Lan}_\alpha = 1_{1_{\mathcal{B}\mathbb{C}}}$. □

Now let $\mathbb{C}$ be a *finitely bicomplete* category $\mathbb{C}$ (i.e., a category with finite limits and finite colimits) and $\alpha \vdash \beta$ an exactness sequent

$$\begin{tikzcd}
\varnothing \arrow[r, "\alpha_1"] \arrow[rrrrrrr, bend right, "\alpha"'] & \mathcal{A}_1 \arrow[r, "\beta_1"] & \mathcal{B}_1 \arrow[r, "\alpha_2"] & \cdots \arrow[r, "\alpha_n"] & \mathcal{A}_n \arrow[r, "\beta_n"] & \mathcal{B}_n \arrow[r, "\omega"] & \mathcal{A} \arrow[r, "\beta"] & \mathcal{B}
\end{tikzcd}$$

as described in (3) together with the notation introduced there. In particular, we consider the finite diagram $D_Y\colon \mathcal{H}_Y \to \mathcal{A}_1 = U(\mathbb{A}_1)$ given by condition (Ax 5). As explained in the beginning of this section, the functor $(D_Y)_{\mathbb{C}}$ has a left adjoint $\mathrm{Lan}_{D_Y}\colon \mathcal{H}_Y\mathbb{C} \to \mathcal{A}_1\mathbb{C}$. Moreover, by Lemma 2.1 and conditions (Ax 1) and (Ax 2), each of the functors $(\alpha_2)_{\mathbb{C}}, \dots, (\alpha_n)_{\mathbb{C}}$ has a left-adjoint-right-inverse. As explained in Section 1, the functors $(\beta_1)_{\mathbb{C}}, \dots, (\beta_n)_{\mathbb{C}}$ are equivalences and their pseudo-inverses $(\beta_i)^{-1}_{\mathbb{C}}$ are chosen to be strict right inverses. Composing all these left adjoints together, we get a left adjoint

$$\mathrm{Lan}_{D_Y^{\mathcal{B}_n}}\colon \mathcal{H}_Y\mathbb{C} \to \mathcal{B}_n\mathbb{C}$$

of $(D_Y^{\mathcal{B}_n})_{\mathbb{C}}$ where $D_Y^{\mathcal{B}_n} = \beta_n \circ \alpha_n \circ \cdots \circ \beta_1 \circ D_Y$ as in condition (Ax 5). Concretely, given a diagram $E\colon \mathcal{H}_Y \to \mathbb{C}$, the left Kan extension $\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} E$ of E along $D_Y^{\mathcal{B}_n}$ is constructed as follows. Firstly, we construct the left Kan extension $L_1 = \mathrm{Lan}_{D_Y} E$ of E along D_Y together with the universal morphism $\lambda^E\colon E \to L_1 \circ D_Y$ using the pointwise Kan extension formula. We then consider the extension $\overline{L_1}$ of L_1 as a $\mathcal{B}_1$-structure using the algorithmic construction described by condition (Ax 3). Suppose by induction that, for some $1 \leqslant i < n$, we have constructed an extension $\overline{L_i}$ of L_1 as a $\mathcal{B}_i$-structure such that $\overline{L_i} \circ \beta_i \circ \alpha_i \circ \cdots \circ \beta_1 = L_1$. The $\mathscr{G}(\mathcal{B}_i)$-structure $\overline{L_i} \circ i^{\mathcal{B}_i}$ can be Kan extended along $\alpha'_{i+1}\colon \mathscr{G}(\mathcal{B}_i) \to U(\mathbb{A}_{i+1})$ to a functor $L'_{i+1}\colon \mathbb{A}_{i+1} \to \mathbb{C}$. In view of condition (Ax 2) and the pointwise Kan extension formula, we can construct L'_{i+1} such that $L'_{i+1} \circ \alpha'_{i+1} = \overline{L_i} \circ i^{\mathcal{B}_i}$. Since $\overline{L_i}$ is a $\mathcal{B}_i$-structure and in view of condition (Ax 1), L'_{i+1} induces an $\mathcal{A}_{i+1}$-structure L_{i+1} such that $L_{i+1} \circ \alpha_{i+1} = \overline{L_i}$. We then construct its extension $\overline{L_{i+1}}$ as a $\mathcal{B}_{i+1}$-structure such that $\overline{L_{i+1}} \circ \beta_{i+1} = L_{i+1}$ using the algorithm provided by condition (Ax 3). This completes the induction. We then set $\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} E = \overline{L_n}$ and the universal morphism is given by

$$\lambda^E\colon E \to L_1 \circ D_Y = \overline{L_n} \circ D_Y^{\mathcal{B}_n}$$

which is the E-component of the unit $\lambda\colon 1_{\mathcal{H}_Y\mathbb{C}} \to (D_Y^{\mathcal{B}_n})_{\mathbb{C}} \circ \mathrm{Lan}_{D_Y^{\mathcal{B}_n}}$ of the adjunction $\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \dashv (D_Y^{\mathcal{B}_n})_{\mathbb{C}}$.

3 Validity in finitely bicomplete categories

For a graph $\mathcal{H}$ and an object C in a category $\mathbb{C}$, we denote by $\Delta_C\colon \mathcal{H} \to \mathbb{C}$ the constant diagram mapping each object of $\mathcal{H}$ to C and each arrow of $\mathcal{H}$ to the identity on C.

Let us now fix a finitely bicomplete category $\mathbb{C}$ and $\alpha \vdash \beta$ an exactness sequent

$$\begin{tikzcd} \varnothing \arrow[r, "\alpha_1"] \arrow[rrrrrrr, bend right, "\alpha"'] & \mathcal{A}_1 \arrow[r, "\beta_1"] & \mathcal{B}_1 \arrow[r, "\alpha_2"] & \cdots \arrow[r, "\alpha_n"] & \mathcal{A}_n \arrow[r, "\beta_n"] & \mathcal{B}_n \arrow[rr, "\omega"] & & \mathcal{A} \arrow[r, "\beta"] & \mathcal{B} \end{tikzcd}$$

as described in (3). Condition (Ax 5) tells us that

$$(Y, (p_H^Y)_H) \equiv_{\mathcal{B}_n} \mathbf{limit}(\mathcal{H}_Y, D_Y^{\mathcal{B}_n}).$$

Therefore, for any object C in $\mathbb{C}$,

$$((\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_C)(Y), ((\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_C)(p_H^Y))_H)$$

is the limit of

$$(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_C) \circ D_Y^{\mathcal{B}_n}.$$

The universal morphism

$$\lambda^{\Delta_C}\colon \Delta_C \to (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_C) \circ D_Y^{\mathcal{B}_n}$$

thus induces a unique morphism $e_C\colon C \to (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_C)(Y)$ such that

$$(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_C)(p_H^Y) \circ e_C = \lambda_H^{\Delta_C}$$

for each $H \in \mathcal{H}_Y$. Let us consider the following pullback square

$$\begin{tikzcd}
\mathrm{Ap}(C) \arrow[r, "\delta_C"] \arrow[d, "\gamma_C"'] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_C)(X) \arrow[d, "(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_C)(q)"] \\
C \arrow[r, "e_C"'] & (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_C)(Y)
\end{tikzcd}$$

where $q\colon X \to Y$ is the arrow in $\mathcal{B}_n$ given by condition (Ax 4). Each morphism $f\colon C \to C'$ in $\mathbb{C}$ induces a morphism of diagrams $\Delta_f\colon \Delta_C \to \Delta_{C'}$ defined for each $H \in \mathcal{H}_Y$ by $(\Delta_f)_H = f$. Let us prove that the equality

$$e_{C'} \circ f = (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_f)_Y \circ e_C$$

holds by precomposing with each leg $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{C'})(p_H^Y)$ of the limit:

$$\begin{aligned}
&(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{C'})(p_H^Y) \circ e_{C'} \circ f \\
&= \lambda_H^{\Delta_{C'}} \circ f \\
&= \lambda_H^{\Delta_{C'}} \circ (\Delta_f)_H \\
&= ((D_Y^{\mathcal{B}_n})_{\mathbb{C}}(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}}(\Delta_f)))_H \circ \lambda_H^{\Delta_C} \\
&= (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_f)_{D_Y^{\mathcal{B}_n}(H)} \circ \lambda_H^{\Delta_C} \\
&= (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_f)_{D_Y^{\mathcal{B}_n}(H)} \circ (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_C)(p_H^Y) \circ e_C \\
&= (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{C'})(p_H^Y) \circ (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_f)_Y \circ e_C
\end{aligned}$$

We deduce from it that the identities

$$\begin{aligned}
e_{C'} \circ f \circ \gamma_C &= (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_f)_Y \circ e_C \circ \gamma_C \\
&= (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_f)_Y \circ (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_C)(q) \circ \delta_C \\
&= (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{C'})(q) \circ (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_f)_X \circ \delta_C
\end{aligned}$$

hold. Therefore, by the universality of the pullback, there exists a unique morphism $\mathrm{Ap}(f)\colon \mathrm{Ap}(C) \to \mathrm{Ap}(C')$ such that

$$\gamma_{C'} \circ \mathrm{Ap}(f) = f \circ \gamma_C$$

and

$$\delta_{C'} \circ \mathrm{Ap}(f) = (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_f)_X \circ \delta_C.$$

This defines an endofunctor $\mathrm{Ap}\colon \mathbb{C} \to \mathbb{C}$ and a natural transformation $\gamma\colon \mathrm{Ap} \to 1_{\mathbb{C}}$.

Theorem 3.1. *Let $\mathbb{C}$ be a finitely bicomplete category and $\alpha \vdash \beta$ an exactness sequent*

$$\begin{tikzcd}
\varnothing \arrow[r, "\alpha_1"] \arrow[rrrrrr, bend right, "\alpha"'] & \mathcal{A}_1 \arrow[r, "\beta_1"] & \mathcal{B}_1 \arrow[r, "\alpha_2"] & \cdots \arrow[r, "\alpha_n"] & \mathcal{A}_n \arrow[r, "\beta_n"] & \mathcal{B}_n \arrow[r, "\omega"] & \mathcal{A} \arrow[r, "\beta"] & \mathcal{B}
\end{tikzcd}$$

as described in (3). We consider the following statements using the above notation:

1. *$\alpha \vdash_{\mathbb{C}} \beta$;*

2. *for any diagram $E\colon \mathcal{H}_Y \to \mathbb{C}$, the morphism $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} E)(q)$ is a strong epimorphism in $\mathbb{C}$;*

3. *for any object C in $\mathbb{C}$, the morphism $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_C)(q)$ is a strong epimorphism in $\mathbb{C}$;*

4. *for any object C in $\mathbb{C}$, the morphism $\gamma_C\colon \mathrm{Ap}(C) \to C$ is a strong epimorphism in $\mathbb{C}$.*

We always have the implications $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftarrow 4$. Moreover, if $\mathbb{C}$ is regular, we also have the remaining implication $3 \Rightarrow 4$.

Proof. The implications $1 \Rightarrow 2 \Rightarrow 3$ are trivial, so let us prove $3 \vee 4 \Rightarrow 1$. Let $\overline{F_n}$ be any $\mathcal{B}_n$-structure in $\mathbb{C}$. We must show that $\overline{F_n}(q)$ is a strong epimorphism. By condition (Ax 5), we know that $(Y, (p_H^Y)_H) \equiv_{\mathcal{B}_n} \mathbf{limit}(\mathcal{H}_Y, D_Y^{\mathcal{B}_n})$. So $(\overline{F_n}(Y), (\overline{F_n}(p_H^Y))_H)$ is the limit of $\overline{F_n} \circ D_Y^{\mathcal{B}_n}$. The projections $\mu_H = \overline{F_n}(p_H^Y)$ give rise to a morphism of diagrams $\mu\colon \Delta_{\overline{F_n}(Y)} \to \overline{F_n} \circ D_Y^{\mathcal{B}_n}$. By the universal property of the Kan extension, there exists a unique morphism $\nu\colon \mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\overline{F_n}(Y)} \to \overline{F_n}$ such that $(\nu \bullet D_Y^{\mathcal{B}_n}) \circ \lambda^{\Delta_{\overline{F_n}(Y)}} = \mu$. Since, for any $H \in \mathcal{H}_Y$, we have

$$\begin{aligned}
\overline{F_n}(p_H^Y) \circ \nu_Y \circ e_{\overline{F_n}(Y)} &= \nu_{D_Y^{\mathcal{B}_n}(H)} \circ (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\overline{F_n}(Y)})(p_H^Y) \circ e_{\overline{F_n}(Y)} \\
&= \nu_{D_Y^{\mathcal{B}_n}(H)} \circ \lambda_H^{\Delta_{\overline{F_n}(Y)}} \\
&= \mu_H \\
&= \overline{F_n}(p_H^Y)
\end{aligned}$$

and since the family $(\overline{F_n}(p_H^Y))_{H \in \mathcal{H}_Y}$ is jointly monomorphic, we know that $\nu_Y \circ e_{\overline{F_n}(Y)} = 1_{\overline{F_n}(Y)}$. This proves that ν_Y is a split epimorphism, and thus a strong epimorphism. It remains to consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Ap}(\overline{F_n}(Y)) \arrow[r, "\delta_{\overline{F_n}(Y)}"] \arrow[d, "\gamma_{\overline{F_n}(Y)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\overline{F_n}(Y)})(X) \arrow[r, "\nu_X"] \arrow[d, "(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\overline{F_n}(Y)})(q)"] & \overline{F_n}(X) \arrow[d, "\overline{F_n}(q)"] \\
\overline{F_n}(Y) \arrow[r, tail, "e_{\overline{F_n}(Y)}"'] & (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\overline{F_n}(Y)})(Y) \arrow[r, two heads, "\nu_Y"'] & \overline{F_n}(Y)
\end{tikzcd}$$

Under the assumption 3, $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\overline{F_n}(Y)})(q)$ is a strong epimorphism. Therefore, $\overline{F_n}(q) \circ \nu_X$ is a strong epimorphism and thus so is $\overline{F_n}(q)$. Under the assumption 4, $\gamma_{\overline{F_n}(Y)} = \nu_Y \circ e_{\overline{F_n}(Y)} \circ \gamma_{\overline{F_n}(Y)} = \overline{F_n}(q) \circ \nu_X \circ \delta_{\overline{F_n}(Y)}$ is a strong epimorphism, thus so is $\overline{F_n}(q)$. Finally, the implication $3 \Rightarrow 4$ holds in the regular context since in that case, strong epimorphisms are stable under pullbacks. □

Remark 3.2. In the case where procedure (f) of condition (Ax 3) for $\alpha \vdash \beta$ is not used, the assumption about the existence of colimits in Theorem 3.1 can be relaxed to the mere existence of the colimits used in the construction of the left adjoint $\mathrm{Lan}_{D_Y^{\mathcal{B}_n}}$. In particular, the equivalence $1 \Leftrightarrow 3$ in the finitely complete context and the equivalence $1 \Leftrightarrow 4$ in the regular context just require the existence of the colimits in $\mathbb{C}$ used in the construction of $\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_C$ for each object $C \in \mathbb{C}$.

4 Validity in varieties

We now treat the case where $\mathbb{C} = \mathbb{V}$ is a *variety of universal algebras*, i.e., a finitary single-sorted algebraic category. We denote by $\mathrm{Fr}(\{\star\})$ the free algebra in $\mathbb{V}$ over one generator $\star$.

Theorem 4.1. *Let $\mathbb{V}$ be a variety of universal algebras. Let also $\alpha \vdash \beta$ be an exactness sequent*

$$\begin{tikzcd} \varnothing \arrow[r, "\alpha_1"] \arrow[rrrrrrr, bend right, "\alpha"'] & \mathcal{A}_1 \arrow[r, "\beta_1"] & \mathcal{B}_1 \arrow[r, "\alpha_2"] & \cdots \arrow[r, "\alpha_n"] & \mathcal{A}_n \arrow[r, "\beta_n"] & \mathcal{B}_n \arrow[r, "\omega"] & \mathcal{A} \arrow[r, "\beta"] & \mathcal{B} \end{tikzcd}$$

as described in (3). Using the notation introduced above, the following statements are equivalent:

1. $\alpha \vdash_{\mathbb{V}} \beta$*;*
2. *for any diagram $E \colon \mathcal{H}_Y \to \mathbb{V}$, the morphism $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} E)(q)$ is surjective;*
3. *for any $\mathbb{V}$-algebra A, the morphism $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_A)(q)$ is surjective;*
4. *for any $\mathbb{V}$-algebra A, the morphism $\gamma_A \colon \mathrm{Ap}(A) \to A$ is surjective;*
5. *the morphism $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\mathrm{Fr}(\{\star\})})(q)$ is surjective;*
6. *the morphism $\gamma_{\mathrm{Fr}(\{\star\})} \colon \mathrm{Ap}(\mathrm{Fr}(\{\star\})) \to \mathrm{Fr}(\{\star\})$ is surjective;*
7. *the element $e_{\mathrm{Fr}(\{\star\})}(\star)$ is in the image of $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\mathrm{Fr}(\{\star\})})(q)$;*
8. *the element $\star$ is in the image of $\gamma_{\mathrm{Fr}(\{\star\})} \colon \mathrm{Ap}(\mathrm{Fr}(\{\star\})) \to \mathrm{Fr}(\{\star\})$.*

Proof. The equivalences $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4$ follow directly from Theorem 3.1 since all varieties of universal algebras are regular. The implication $3 \Rightarrow 5$ is trivial. The implication $5 \Rightarrow 6$ follows from the fact that regular epimorphisms are stable under pullbacks in any regular category. The implication $6 \Rightarrow 8$ is trivial and the equivalence $7 \Leftrightarrow 8$ follows immediately from the description of pullbacks in a variety. It remains to prove $8 \Rightarrow 4$. Given any $\mathbb{V}$-algebra A and any element $a \in A$, we must show that a is in the image of γ_A. We consider the unique homomorphism $f \colon \mathrm{Fr}(\{\star\}) \to A$ such that $f(\star) = a$. Since the following diagram commutes by naturality of γ,

$$\begin{tikzcd} \mathrm{Ap}(\mathrm{Fr}(\{\star\})) \arrow[r, "\mathrm{Ap}(f)"] \arrow[d, "\gamma_{\mathrm{Fr}(\{\star\})}"'] & \mathrm{Ap}(A) \arrow[d, "\gamma_A"] \\ \mathrm{Fr}(\{\star\}) \arrow[r, "f"'] & A \end{tikzcd}$$

and since $\star$ is in the image of $\gamma_{\mathrm{Fr}(\{\star\})}$ by assumption, we know that a is in the image of $\gamma_A \circ \mathrm{Ap}(f)$ and thus in the image of γ_A. $\square$

5 Examples

In the display of the particular sketches below, we are going to use the simplification rules introduced in [23]. In particular, we display the convergence conditions by the limits and colimits they represent. Moreover, in the display of graphs containing the underlying graph of a category as a subgraph, we may omit to represent from this subgraph identity arrows and composite of arrows which are already displayed.

5.1 Protomodularity

As a first example, let us study the case of protomodularity [4]. As shown in [6], a finitely complete category $\mathbb{C}$ is *protomodular* if and only if, for any diagram

$$\begin{tikzcd} & Y \arrow[d, shift right, "p"'] \\ V \arrow[r, "f"'] & W \arrow[u, shift right, "s"'] \end{tikzcd} \tag{4}$$

where $p \circ s = 1_W$, when considering the pullback (P, p', f') of f along p,

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, shift right, "p"'] \\ V \arrow[r, "f"'] & W \arrow[u, shift right, "s"'] \end{tikzcd}$$

the morphisms f' and s are jointly strongly epimorphic. This happens exactly when, for any further extension of the above diagram to

$$\begin{tikzcd} & & X \arrow[dl, "q"] \\ P \arrow[r, "f'"] \arrow[d, "p'"'] \arrow[urr, bend left, "g"] & Y \arrow[d, shift right, "p"'] & \\ V \arrow[r, "f"'] & W \arrow[u, shift right, "s"'] \arrow[uur, bend right, "h"'] & \end{tikzcd} \tag{5}$$

where $q \circ g = f'$ and $q \circ h = s$, the morphism q is a strong epimorphism. This latter property can be stated as the property $\alpha \vdash_{\mathbb{C}} \beta$ for an exactness sequent $\alpha \vdash \beta$ as described in (3) as follows. Firstly, let $\mathbb{A}_1$ be the path category associated to the graph displayed in (4) and equipped with the condition $p \cdot s = 1_W$. By Lemma 3.2 in [25], we know that $\mathbb{A}_1$ is a finite category. We let $\mathcal{A}_1$ be the underlying sketch of $\mathbb{A}_1$. We then let $\mathcal{B}_1 \supseteq \mathcal{A}_1$ be the sketch represented by

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "p'"'] & Y \arrow[d, shift right, "p"'] \\ V \arrow[r, "f"'] & W \arrow[u, shift right, "s"'] \end{tikzcd}$$

conditions from $\mathcal{A}_1$ together with:
(P, p', f') represents the pullback of f along p.

Let also $\mathbb{A}_2$ be the path category associated to the graph displayed in (5) and equipped with the conditions $p \cdot s = 1_W$, $f \cdot p' = p \cdot f'$, $q \cdot g = f'$ and $q \cdot h = s$. It follows again from Lemma 3.2 in [25] that $\mathbb{A}_2$ is a finite category. We then let $\mathcal{A}_2 = \mathcal{B}_2 \supseteq \mathcal{B}_1$ be the underlying sketch of $\mathbb{A}_2$ further equipped with the commutativity and convergence conditions from $\mathcal{B}_1$.

As prescribed by condition (Ax 4), we define $\mathcal{A} \supseteq \mathcal{B}_2$ as the sketch represented by

$$\begin{tikzcd}
 & Y' \arrow[loop left, "i_{Y'}"] \arrow[d, "m_q"] & X \arrow[l, "l_q"'] \arrow[dl, "q"] \\
P \arrow[urr, "g"] \arrow[r, "f'"] \arrow[d, "p'"'] & Y \arrow[d, shift right, "p"'] & \\
V \arrow[r, "f"'] & W \arrow[u, shift right, "s"'] \arrow[uur, bend right, "h"'] &
\end{tikzcd}$$

conditions from $\mathcal{B}_2$ together with:
$q = m_q \cdot l_q$
$i_{Y'} = 1_{Y'}$
$(Y', i_{Y'}, i_{Y'})$ represents the kernel pair of m_q.

Again as prescribed by condition (Ax 4), the sketch $\mathcal{B}$ is obtained by adding to $\mathcal{A}$ the condition 'm_q represents an isomorphism'. By construction,

$$\begin{tikzcd}
\varnothing \arrow[r, "\alpha_1"] \arrow[rrrrrr, bend right, "\alpha"'] & \mathcal{A}_1 \arrow[r, "\beta_1"] & \mathcal{B}_1 \arrow[r, "\alpha_2"] & \mathcal{A}_2 \arrow[r, "\beta_2"] & \mathcal{B}_2 \arrow[r, "\omega"] & \mathcal{A} \arrow[r, "\beta"] & \mathcal{B}
\end{tikzcd}$$

satisfies the conditions (Ax 0)–(Ax 5) described in (3). For (Ax 5), one can take $\mathcal{H}_Y$ to be the graph with a single object $*$ and no arrow, $D_Y : \mathcal{H}_Y \to \mathcal{A}_1$ to be defined by $D_Y(*) = Y$ and p_*^Y to be the empty path on Y. As mentioned above, a finitely complete category $\mathbb{C}$ is protomodular if and only if $\alpha \vdash_{\mathbb{C}} \beta$.

Given any object C in a finitely bicomplete category $\mathbb{C}$, the Kan extension $\mathrm{Lan}_{D_Y^{\mathcal{B}_2}} \Delta_C$ of the constant diagram $\Delta_C \colon \mathcal{H}_Y \to \mathbb{C}$, constructed in the general case in Section 2, is described as follows. We first need to compute the left Kan extension $L_1 = \mathrm{Lan}_{D_Y} \Delta_C \colon \mathcal{A}_1 \to \mathbb{C}$ of Δ_C along D_Y. Using the pointwise Kan extension formula, it is given by the $\mathcal{A}_1$-structure

$$\begin{tikzcd}
 & 2C \arrow[d, shift right, "\nabla_C"'] \\
0 \arrow[r, "!_C"'] & C \arrow[u, shift right, "\iota_2"']
\end{tikzcd}$$

in $\mathbb{C}$ where 0 is the initial object, $!_C$ is the unique morphism $0 \to C$, $2C$ is the cosquare of C with coproduct injections ι_1 and ι_2 and ∇_C is the codiagonal. We have obtained $L_1(V) = 0$ since $\mathbb{A}_1(Y, V) = \varnothing$, $L_1(W) = C$ since $\mathbb{A}_1(Y, W) = \{p\}$ and $L_1(Y) = 2C$ since $\mathbb{A}_1(Y, Y) = \{1_Y, s \circ p\}$ and since the three graphs $(D_Y \downarrow A)$ for $A = V, W, Y$ are discrete. The extension of L_1 as a $\mathcal{B}_1$-structure $\overline{L_1}$ is given by computing the pullback of $!_C$ along ∇_C.

$$\begin{tikzcd}
0 \times_{!_C, \nabla_C} 2C \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & 2C \arrow[d, shift right, "\nabla_C"'] \\
0 \arrow[r, "!_C"'] & C \arrow[u, shift right, "\iota_2"']
\end{tikzcd}$$

Since $\beta_2 = 1_{\mathcal{A}_2}$, it remains to compute $L_2(X)$ via the pointwise Kan extension formula.

This gives rise to the following extension $L_2 = \overline{L_2}$ of $\overline{L_1}$

$$\begin{tikzcd}
 & & (0 \times_{!_C, \nabla_C} 2C) + C \arrow[dl, "\left(\begin{smallmatrix} p_2 \\ \iota_2 \end{smallmatrix}\right)"'] \\
0 \times_{!_C, \nabla_C} 2C \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[urr, bend left, "\iota_1'"] \arrow[dr, phantom, "\lrcorner", very near start] & 2C \arrow[d, shift right, "\nabla_C"'] & \\
0 \arrow[r, "!_C"'] & C \arrow[u, shift right, "\iota_2"'] \arrow[uur, bend right, "\iota_2'"'] &
\end{tikzcd}$$

where $(0 \times_{!_C, \nabla_C} 2C) + C$ is the coproduct of $0 \times_{!_C, \nabla_C} 2C$ and C with coproduct injections ι_1' and ι_2' and where $\left(\begin{smallmatrix} p_2 \\ \iota_2 \end{smallmatrix}\right)$ is the morphism induced by p_2 and ι_2. Theorem 3.1 (together with Remark 3.2) tells us that a finitely complete category $\mathbb{C}$ with finite coproducts is protomodular if and only if, for any object C in $\mathbb{C}$, $\left(\begin{smallmatrix} p_2 \\ \iota_2 \end{smallmatrix}\right)$ is a strong epimorphism, i.e., p_2 and ι_2 are jointly strongly epimorphic. This particular case of Theorem 3.1 is contained in Theorem 6.4 of [9]. In addition, if $\mathbb{C}$ is a regular category with finite coproducts, we know from Theorem 3.1 that $\mathbb{C}$ is protomodular if and only if, for any object C in $\mathbb{C}$, the pullback p_1' of $\left(\begin{smallmatrix} p_2 \\ \iota_2 \end{smallmatrix}\right)$ along $e_c = \iota_1$ is a regular epimorphism.

$$\begin{tikzcd}
C \times_{\iota_1, \left(\begin{smallmatrix} p_2 \\ \iota_2 \end{smallmatrix}\right)} ((0 \times_{!_C, \nabla_C} 2C) + C) \arrow[r, "p_2'"] \arrow[d, "p_1'"'] \arrow[dr, phantom, "\lrcorner", very near start] & (0 \times_{!_C, \nabla_C} 2C) + C \arrow[d, "\left(\begin{smallmatrix} p_2 \\ \iota_2 \end{smallmatrix}\right)"] \\
C \arrow[r, "\iota_1"'] & 2C
\end{tikzcd}$$

If we further assume that $\mathbb{C} = \mathbb{V}$ is a variety, we only need to consider the above diagram in the case where C is the free algebra on one generator. We obtain the diagram

$$\begin{tikzcd}
 & & P + \mathrm{Fr}(\{y\}) \arrow[dl, "\left(\begin{smallmatrix} p_2 \\ s \end{smallmatrix}\right)"'] \\
P \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[urr, bend left, "\iota_1'"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fr}(\{x, y\}) \arrow[d, shift right, "p"'] & \\
\mathrm{Fr}(\varnothing) \arrow[r, "!"'] & \mathrm{Fr}(\{y\}) \arrow[u, shift right, "s"'] \arrow[uur, bend right, "\iota_2'"'] &
\end{tikzcd}$$

where p and s are determined by $p(x) = y = p(y)$ and $s(y) = y$. Theorem 4.1 says that $\mathbb{V}$ is protomodular if and only if x is in the image of $\left(\begin{smallmatrix} p_2 \\ s \end{smallmatrix}\right)$. The pullback P is described as

$$P = \{(c, d(x, y)) \in \mathrm{Fr}(\varnothing) \times \mathrm{Fr}(\{x, y\}) \mid c = d(y, y) \text{ is an identity in the theory of } \mathbb{V}\}$$

where c represents a constant term and $d(x, y)$ a binary term. The image of $\left(\begin{smallmatrix} p_2 \\ s \end{smallmatrix}\right)$ in $\mathrm{Fr}(\{x, y\})$ is given by

$$\begin{aligned}
\mathrm{Im}\left(\begin{smallmatrix} p_2 \\ s \end{smallmatrix}\right) = \{&\pi(d_1(x, y), \ldots, d_n(x, y), u_1(y), \ldots, u_m(y)) \mid n, m \geqslant 0,\ \pi \text{ is an } (n+m)\text{-ary term}, \\
&(c_i, d_i(x, y)) \in P \text{ for each } 1 \leqslant i \leqslant n \text{ and } u_i(y) \in \mathrm{Fr}(\{y\}) \text{ for each } 1 \leqslant i \leqslant m\} \\
= \{&\pi'(d_1(x, y), \ldots, d_n(x, y), y) \mid n \geqslant 0,\ \pi' \text{ is an } (n+1)\text{-ary term and} \\
&(c_i, d_i(x, y)) \in P \text{ for each } 1 \leqslant i \leqslant n\}.
\end{aligned}$$

Therefore, Theorem 4.1 gives in this particular case exactly the characterization of protomodular varieties established in [7].

5.2 Involution-rigidness

The involution-rigidness property has been introduced in [32]. A *generalized fixed point* of an involution $i\colon W \to W$ in a category $\mathbb{C}$ is a morphism $p\colon P \to W$ such that $i \circ p = p$. A morphism $f\colon Y \to W$ in $\mathbb{C}$ is said to be *rigid* under the involution $i\colon W \to W$ when $i \circ f$ factors through any monomorphism $m\colon V \rightarrowtail W$ through which f and any generalized fixed point of i factor. A category has the *involution-rigidness property* when every morphism $f\colon Y \to W$ is rigid under any involution $i\colon W \to W$. In the regular context, this is equivalent to requiring that for any morphism $f\colon Y \to W$ and any involution $i\colon W \to W$, considering the equalizer

$$\begin{tikzcd} E \arrow[r, "e"] & W \arrow[r, shift left, "i"] \arrow[r, shift right, "1_W"'] & W, \end{tikzcd}$$

and any factorizations $f = g \circ f'$ and $e = g \circ e'$ of f and e through a common morphism $g\colon V \to W$ and taking the pullback X of g along $i \circ f$,

$$\begin{tikzcd} X \arrow[r, "r"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, "g"] \\ Y \arrow[r, "i \circ f"'] & W \end{tikzcd}$$

the morphism q is a strong epimorphism. This property can be stated as the property $\alpha \vdash_{\mathbb{C}} \beta$ for an exactness sequent $\alpha \vdash \beta$ as described in (3). Let $\mathcal{A}_1$ be the underlying sketch of the path category of the graph

$$\begin{tikzcd} Y \arrow[r, "f"] & W \arrow[loop above, "i"] \end{tikzcd}$$

equipped with the condition $i \cdot i = 1_W$. We then let $\mathcal{B}_1 \supseteq \mathcal{A}_1$ be the sketch represented by

$$\begin{tikzcd} Y \arrow[r, "f"] & W \arrow[loop above, "i"] & E \arrow[l, "e"'] \end{tikzcd}$$

conditions from $\mathcal{A}_1$ together with:
(E, e) represents the equalizer of i and 1_W.

Now let $\mathbb{A}_2$ be the path category of the graph

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "f'"'] & W \arrow[loop above, "i"] & E \arrow[l, "e"'] \arrow[dl, "e'"] \\ & V \arrow[u, "g"] & \end{tikzcd}$$

equipped with the conditions $i \cdot i = 1_W$, $f = g \cdot f'$ and $e = g \cdot e'$. By Lemma 3.2 in [25], $\mathbb{A}_2$ is a finite category. Let $\mathcal{A}_2 \supseteq \mathcal{B}_1$ be the underlying sketch of $\mathbb{A}_2$ further equipped with the commutativity and convergence conditions from $\mathcal{B}_1$. Let $\mathcal{B}_2 \supseteq \mathcal{A}_2$ be the sketch represented by

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "f'"'] & W \arrow[loop above, "i"] & E \arrow[l, "e"'] \arrow[dl, "e'"] \\ X \arrow[u, "q"] \arrow[r, "r"'] & V \arrow[u, "g"] & \end{tikzcd}$$

conditions from $\mathcal{A}_2$ together with:
(X, q, r) represents the pullback of $i \circ f$ along g.

Finally, we define $\mathcal{A}$ and $\mathcal{B}$ as prescribed by condition (Ax 4) for the arrow $q\colon X \to Y$ of $\mathcal{B}_2$. We thus have an exactness sequent

$$\begin{tikzcd}
\varnothing \arrow[r, "\alpha_1"] \arrow[rrrrr, bend right, "\alpha"'] & \mathcal{A}_1 \arrow[r, "\beta_1"] & \mathcal{B}_1 \arrow[r, "\alpha_2"] & \mathcal{A}_2 \arrow[r, "\beta_2"] & \mathcal{B}_2 \arrow[r, "\omega"] & \mathcal{A} \arrow[r, "\beta"] & \mathcal{B}
\end{tikzcd}$$

satisfying conditions (Ax 0)–(Ax 5) and such that, for any regular category $\mathbb{C}$, $\alpha \vdash_{\mathbb{C}} \beta$ holds if and only if $\mathbb{C}$ has the involution-rigidness property. For (Ax 5), we consider $\mathcal{H}_Y$ to be the graph with a single object $*$ and no arrow, $D_Y\colon \mathcal{H}_Y \to \mathcal{A}_1$ to be defined by $D_Y(*) = Y$ and p_*^Y to be the empty path on Y.

Let $\mathbb{C}$ be a regular category with finite colimits. Given any object C in $\mathbb{C}$, the Kan extension $\mathrm{Lan}_{D_Y^{\mathcal{B}_2}} \Delta_C$ of the constant diagram $\Delta_C\colon \mathcal{H}_Y \to \mathbb{C}$, constructed in the general case in Section 2, is described as follows. We first need to compute the left Kan extension $L_1 = \mathrm{Lan}_{D_Y} \Delta_C \colon \mathcal{A}_1 \to \mathbb{C}$ of Δ_C along D_Y. Using the pointwise Kan extension formula, it is given by the following $\mathcal{A}_1$-structure.

$$\begin{tikzcd}
C \arrow[r, "\iota_2"] & 2C \arrow[loop above, "\binom{\iota_2}{\iota_1}"]
\end{tikzcd}$$

To describe its extension $\overline{L_1}$ as a $\mathcal{B}_1$-structure, we only need to compute the equalizer $e\colon E \to 2C$ of $\binom{\iota_2}{\iota_1}$ and the identity on $2C$.

$$\begin{tikzcd}
C \arrow[r, "\iota_2"] & 2C \arrow[loop above, "\binom{\iota_2}{\iota_1}"] & E \arrow[l, "e"']
\end{tikzcd}$$

We then need to compute its extension L_2 as an $\mathcal{A}_2$-structure via the pointwise Kan extension formula. We obtain the $\mathcal{A}_2$-structure represented by:

$$\begin{tikzcd}
C \arrow[r, "\iota_2"] \arrow[dr, "\iota_1'"'] & 2C \arrow[loop above, "\binom{\iota_2}{\iota_1}"] & E \arrow[l, "e"'] \arrow[dl, "\iota_2'"] \\
& C+E \arrow[u, "\binom{\iota_2}{e}"] &
\end{tikzcd}$$

Finally, the $\mathcal{B}_2$-structure $\overline{L_2} = \mathrm{Lan}_{D_Y^{\mathcal{B}_2}} \Delta_C$ is obtained by computing the pullback P of $\binom{\iota_2}{e}$ along $\binom{\iota_2}{\iota_1} \circ \iota_2 = \iota_1$.

$$\begin{tikzcd}
C \arrow[r, "\iota_2"] \arrow[dr, "\iota_1'"] & 2C \arrow[loop above, "\binom{\iota_2}{\iota_1}"] & E \arrow[l, "e"'] \arrow[dl, "\iota_2'"] \\
P \arrow[u, "p_2"] \arrow[r, "p_1"'] & C+E \arrow[u, "\binom{\iota_2}{e}"] &
\end{tikzcd}$$

Theorem 3.1 says that a regular category $\mathbb{C}$ with binary coproducts has the involution-rigidness property if and only if, for any object C in $\mathbb{C}$, the morphism $p_2\colon P \to C$ as constructed above is a regular epimorphism.

If we further assume that $\mathbb{C} = \mathbb{V}$ is a variety, we only need to consider the above diagram in the case where $C = \mathrm{Fr}(\{x\})$ is the free algebra on one generator. We then get $2C = \mathrm{Fr}(\{x_1, x_2\})$ and

$$E = \{a(x_1, x_2) \in \mathrm{Fr}(\{x_1, x_2\}) \,|\, a(x_2, x_1) = a(x_1, x_2)\}.$$

The image of p_2 in $\mathrm{Fr}(\{x\})$ can be described as

$$\begin{aligned}\mathrm{Im}(p_2) = \{u(x) \in \mathrm{Fr}(\{x\}) \,|\, &\exists\, n, m \geqslant 0, \pi \text{ an } (n+m)\text{-ary term}, u_1(x), \dots, u_n(x) \in \mathrm{Fr}(\{x\}) \\ &\text{and } a_1(x_1, x_2), \dots, a_m(x_1, x_2) \in E \text{ such that} \\ &\pi(u_1(x_2), \dots, u_n(x_2), a_1(x_1, x_2), \dots, a_m(x_1, x_2)) = u(x_1)\} \\ = \{u(x) \in \mathrm{Fr}(\{x\}) \,|\, &\exists\, m \geqslant 0, \pi' \text{ a } (1+m)\text{-ary term} \\ &\text{and } a_1(x_1, x_2), \dots, a_m(x_1, x_2) \in E \\ &\text{such that } \pi'(x_2, a_1(x_1, x_2), \dots, a_m(x_1, x_2)) = u(x_1)\}.\end{aligned}$$

Theorem 4.1 tells us that $\mathbb{V}$ has the involution-rigidness property if and only if $x \in \mathrm{Im}(p_2)$, which is the characterization obtained in [32].

5.3 The Mal'tsev property

Mal'tsev categories have been introduced in [12] as finitely complete categories in which every binary relation is difunctional. In a regular context, this is equivalent to the condition that for each diagram

$$\begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[dr, "r_2"] & \\ A & & B \\ & Y \arrow[ul, shift left, "a_2"] \arrow[ul, shift right, "a_1"'] \arrow[uu, shift right=2, "y_1"] \arrow[uu, "y_2"] \arrow[uu, shift left=2, "y_3"'] \arrow[ur, shift left, "b_1"] \arrow[ur, shift right, "b_2"'] & \end{tikzcd}$$

where $r_1 \circ y_1 = a_1$, $r_2 \circ y_1 = b_1$, $r_1 \circ y_2 = a_1$, $r_2 \circ y_2 = b_2$, $r_1 \circ y_3 = a_2$ and $r_2 \circ y_3 = b_2$, considering the product of A and B and the following pullback,

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "(r_1, r_2)"] \\ Y \arrow[r, "(a_2, b_1)"'] & A \times B \end{tikzcd}$$

the morphism $q\colon X \to Y$ is a strong epimorphism. This formulation of the Mal'tsev property on a regular category $\mathbb{C}$ can be expressed as the condition $\alpha \vdash_{\mathbb{C}} \beta$ for an exactness sequent $\alpha \vdash \beta$ as described in (3) with $n = 1$. Theorem 3.1 then tells us that a regular category $\mathbb{C}$ with finite copowers is a Mal'tsev category if and only if, for any object $C \in \mathbb{C}$, the projection p_2 in the pullback

$$\begin{tikzcd} P \arrow[r, "p_1"] \arrow[d, "p_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & 3C \arrow[d, "\left(\begin{smallmatrix} \iota_1 & \iota_1 \\ \iota_1 & \iota_2 \\ \iota_2 & \iota_2 \end{smallmatrix}\right)"] \\ C \arrow[r, "(\iota_2, \iota_1)"'] & (2C)^2 \end{tikzcd}$$

is a strong epimorphism. This characterization has been established in [8]. Theorem 4.1 says in this particular case that a variety $\mathbb{V}$ is a Mal'tsev category if and only if there exists a ternary term $p(x,y,z)$ satisfying the identities $p(x,x,y)=y$ and $p(x,y,y)=x$ as established in [34].

In the regular context, the Mal'tsev property is thus a property of the form $\alpha \vdash_{\mathbb{C}} \beta$ for an exactness sequent $\alpha \vdash \beta$ as described in (3) such that, moreover, $n=1$ and the procedures (b), (f) and (g) have not been used in condition (Ax 3). This kind of exactness properties in the regular context is the subject of [24] where it is proved that these are exactly the finite conjunctions of matrix conditions as introduced in [29] (in the theory Th[**Set**] of sets). See Subsection 6.3 for more details.

Let us now study another exactness sequent describing the Mal'tsev property. It has been shown in [5] that a finitely complete category $\mathbb{C}$ is a Mal'tsev category if and only if for each diagram

$$\begin{tikzcd} & B \arrow[d, two heads, shift right, "g"'] \\ A \arrow[r, two heads, shift left, "f"] & C \arrow[l, tail, shift left, "s"] \arrow[u, tail, shift right, "t"'] \end{tikzcd}$$

where $f\circ s = 1_C = g\circ t$, considering the pullback (Y,p_1,p_2) of f along g, the morphisms $l\colon A\to Y$ and $r\colon B\to Y$ determined by $p_1\circ l = 1_A$, $p_2\circ l = t\circ f$, $p_1\circ r = s\circ g$ and $p_2\circ r = 1_B$ are jointly strongly epimorphic, i.e., for any extension of the diagram as

$$\begin{tikzcd} X \arrow[dr, "q"] & & \\ & Y \arrow[r, two heads, shift left, "p_2"] \arrow[d, two heads, shift right, "p_1"'] & B \arrow[l, tail, shift left, "r"] \arrow[d, two heads, shift right, "g"'] \arrow[ull, bend right, "k"'] \\ & A \arrow[r, two heads, shift left, "f"] \arrow[u, tail, shift right, "l"'] \arrow[uul, bend left, "h"] & C \arrow[l, tail, shift left, "s"] \arrow[u, tail, shift right, "t"'] \end{tikzcd}$$

with $q\circ h = l$ and $q\circ k = r$, the morphism $q\colon X\to Y$ is a strong epimorphism. It is not hard to construct an exactness sequent

$$\begin{tikzcd} \varnothing \arrow[r, "\alpha_1"] \arrow[rrrrr, bend right, "\alpha"'] & \mathcal{A}_1 \arrow[r, "\beta_1"] & \mathcal{B}_1 \arrow[r, "\alpha_2"] & \mathcal{A}_2 \arrow[r, "\beta_2"] & \mathcal{B}_2 \arrow[r, "\omega"] & \mathcal{A} \arrow[r, "\beta"] & \mathcal{B} \end{tikzcd}$$

as in (3) which describes this property, i.e., such that $\alpha \vdash_{\mathbb{C}} \beta$ holds for a finitely complete category $\mathbb{C}$ if and only if $\mathbb{C}$ is a Mal'tsev category. Theorem 3.1 applied to this particular exactness sequent gives that a finitely complete category $\mathbb{C}$ with binary copowers is a Mal'tsev category if and only if, for any object $C\in\mathbb{C}$, considering the kernel pair $\mathrm{Eq}[\nabla_C]$ of $\nabla_C = \begin{pmatrix} 1_C \\ 1_C \end{pmatrix} : 2C\to C$, the morphism $w = \begin{pmatrix} \iota_1 & \iota_2 \\ \iota_2 & \iota_2 \\ \iota_2 & \iota_1 \\ \iota_2 & \iota_2 \end{pmatrix} : 4C \to \mathrm{Eq}[\nabla_C]$ is a strong epimorphism.

$$\begin{tikzcd}
4C \arrow[dr, "w"] & & \\
& \mathrm{Eq}[\nabla_C] \arrow[r, shift left, two heads, "p_2"] \arrow[d, shift right, two heads, "p_1"'] & 2C \arrow[l, shift left, "r"] \arrow[d, shift right, two heads, "\nabla_C"'] \arrow[ull, bend right, "\binom{\iota_3'}{\iota_4'}"'] \\
& 2C \arrow[u, shift right, "l"'] \arrow[r, shift left, two heads, "\nabla_C"] \arrow[uul, bend left, "\binom{\iota_1'}{\iota_2'}"] & C \arrow[l, shift left, "\iota_2"] \arrow[u, shift right, "\iota_2"']
\end{tikzcd}$$

Moreover, if $\mathbb{C} = \mathbb{V}$ is a variety, we only need to consider this diagram when $C = \mathrm{Fr}(\{x\})$ is the free algebra on one generator. In this case, $\mathrm{Eq}[\nabla_{\mathrm{Fr}(\{x\})}]$ can be described as

$$\mathrm{Eq}[\nabla_{\mathrm{Fr}(\{x\})}] = \{(a(x_1, x_2), b(x_1, x_2)) \in \mathrm{Fr}(\{x_1, x_2\})^2 \mid a(x, x) = b(x, x)\}$$

and the morphism $w\colon \mathrm{Fr}(\{y_1, y_2, y_3, y_4\}) \to \mathrm{Eq}[\nabla_{\mathrm{Fr}(\{x\})}]$ is determined by $w(y_1) = (x_1, x_2)$, $w(y_2) = (x_2, x_2)$, $w(y_3) = (x_2, x_1)$ and $w(y_4) = (x_2, x_2)$. Theorem 4.1 says in this case that $\mathbb{V}$ is a Mal'tsev category if and only if $e_{\mathrm{Fr}(\{x\})}(x) = (x_1, x_1) \in \mathrm{Eq}[\nabla_{\mathrm{Fr}(\{x\})}]$ is in the image of w, that is if and only if there exists a quaternary term $t(y_1, y_2, y_3, y_4)$ such that $t(x_1, x_2, x_2, x_2) = x_1$ and $t(x_2, x_2, x_1, x_2) = x_1$ are theorems of the theory of $\mathbb{V}$. This is of course equivalent to the well-known characterization of Mal'tsev varieties; to see this, given such a quaternary term $t(y_1, y_2, y_3, y_4)$, one can construct a Mal'tsev term as $p(x, y, z) = t(x, y, z, y)$ and given a Mal'tsev term $p(x, y, z)$, one can construct such a quaternary term as $t(y_1, y_2, y_3, y_4) = p(y_1, y_2, y_3)$.

6 The pointed context

One can also consider the pointed version of the class of exactness sequents described in (3). We consider here exactness sequents $\alpha \vdash \beta$ that can be decomposed as sequences of subsketch inclusions as in

$$\begin{tikzcd}
\varnothing \arrow[r, "\beta_0"] \arrow[rrrrrrr, bend right=15, "\alpha"'] & \mathcal{B}_0 \arrow[r, "\alpha_1"] & \mathcal{A}_1 \arrow[r, "\beta_1"] & \mathcal{B}_1 \arrow[r, "\alpha_2"] & \cdots \arrow[r, "\alpha_n"] & \mathcal{A}_n \arrow[r, "\beta_n"] & \mathcal{B}_n \arrow[r, "\omega"] & \mathcal{A} \arrow[r, "\beta"] & \mathcal{B}
\end{tikzcd} \tag{6}$$

satisfying conditions (Ax 0 *), (Ax 1), (Ax 2), (Ax 3), (Ax 4) and (Ax 5) where (Ax 1)–(Ax 5) are as in (3) and (Ax 0 *) is now:

(Ax 0 *) $n \geqslant 1$ is a natural number, $\alpha = \omega \circ \beta_n \circ \alpha_n \circ \cdots \circ \beta_1 \circ \alpha_1 \circ \beta_0$ and $\mathcal{B}_0$ is the sketch with a single object Z, no arrows, no commutativity conditions and

$$(Z, \varnothing) = \mathbf{limit}(\varnothing, D_!) \quad \text{and} \quad (Z, \varnothing) = \mathbf{colimit}(\varnothing, D_!)$$

as convergence conditions where $D_!$ is the unique diagram $\varnothing \to \mathcal{B}_0$.

In other words, the convergence conditions in $\mathcal{B}_0$ just say that Z represents a terminal and initial object, i.e., a zero object. Therefore, all the sketches $\mathcal{A}_1$, $\mathcal{B}_1$, …, $\mathcal{A}_n$ and $\mathcal{B}_n$ contain the object Z with the conditions saying it represents a zero object.

If $\mathbb{C}$ is a pointed category with finite limits, the condition $\alpha \vdash_{\mathbb{C}} \beta$ means as before: 'for any $\mathcal{B}_n$-structure $\overline{F_n}$ in $\mathbb{C}$, the morphism $\overline{F_n}(q)$ is a strong epimorphism'. Moreover, if

$\mathbb{C}$ is a finitely bicomplete pointed category, the functors $(\beta_1)_{\mathbb{C}},\ldots,(\beta_n)_{\mathbb{C}}$ are equivalences of categories. The condition $\alpha \vdash_{\mathbb{C}} \beta$ can then be reformulated in this case as: 'for any $\mathcal{A}_1$-structure F_1 in $\mathbb{C}$, considering its extension $\overline{F_1}$ as a $\mathcal{B}_1$-structure, for any extension of it as an $\mathcal{A}_2$-structure F_2, considering its extension $\overline{F_2}$ as a $\mathcal{B}_2$-structure, ..., for any extension of $\overline{F_{n-1}}$ as an $\mathcal{A}_n$-structure F_n, considering its extension $\overline{F_n}$ as a $\mathcal{B}_n$-structure, then $\overline{F_n}(q)$ is a strong epimorphism'.

Let $\mathbb{C}$ be a finitely bicomplete pointed category and let us consider the diagram $D_Y : \mathcal{H}_Y \to \mathcal{A}_1$ given by condition (Ax 5). Viewing $\mathcal{H}_Y$ as a sketch with no conditions, the functor $(D_Y)_{\mathbb{C}}$ has a left adjoint $\mathrm{Lan}_{D_Y} : \mathcal{H}_Y\mathbb{C} \to \mathcal{A}_1\mathbb{C}$. Given any diagram $E : \mathcal{H}_Y \to \mathbb{C}$, the left Kan extension $\mathrm{Lan}_{D_Y} E : \mathcal{A}_1 \to \mathbb{C}$ of E along D_Y

$$\begin{tikzcd} \mathcal{H}_Y \arrow[r, "D_Y"] \arrow[dr, "E"'] & \mathcal{A}_1 \arrow[d, "\mathrm{Lan}_{D_Y} E"] \\ & \mathbb{C} \arrow[u, phantom, "\lambda^E \nearrow", near start, shift right=3] \end{tikzcd}$$

can be computed via the pointwise pointed left Kan extension formula (see e.g. [13]): For an object $A \in \mathcal{A}_1$, let $(D \downarrow A)^*$ be the graph obtained from the graph $(D \downarrow A)$ by adding an object 1 and arrows $!_{(H,f)} : (H, f) \to 1$ for any $(H, f) \in (D \downarrow A)$ such that $f : D(H) \to A$ factors through $Z \in \mathcal{B}_0$. We denote by $\psi_A : (D \downarrow A)^* \to \mathbb{C}$ the diagram defined on objects by $\psi_A(H, f) = E(H)$ and $\psi_A(1) = 0$, the zero object of $\mathbb{C}$, and on arrows by $\psi_A(h) = E(h)$ and $\psi_A(!_{(H,f)})$ is the unique morphism $E(H) \to 0$. The object $(\mathrm{Lan}_{D_Y} E)(A)$ is the colimit of this diagram ψ_A and we denote by $s^A_{(H,f)} : E(H) \to (\mathrm{Lan}_{D_Y} E)(A)$ and $s^A_1 : 0 \to (\mathrm{Lan}_{D_Y} E)(A)$ the colimit legs. Given an arrow $a : A \to A'$ in $\mathcal{A}_1$, the morphism $(\mathrm{Lan}_{D_Y} E)(a)$ is the unique morphism such that $(\mathrm{Lan}_{D_Y} E)(a) \circ s^A_{(H,f)} = s^{A'}_{(H,a\circ f)}$ for any $(H, f) \in (D \downarrow A)$. The E-component λ^E of the unit is determined by $\lambda^E_H = s^{D(H)}_{(H,1_{D(H)})}$ for any object $H \in \mathcal{H}$. Using now the left-adjoint-right-inverses of $(\alpha_2)_{\mathbb{C}}, \ldots, (\alpha_n)_{\mathbb{C}}$ given by Lemma 2.1 and the pseudo-inverse-strict-right-inverses of $(\beta_1)_{\mathbb{C}}, \ldots, (\beta_n)_{\mathbb{C}}$, we obtain a left adjoint

$$\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} : \mathcal{H}_Y\mathbb{C} \to \mathcal{B}_n\mathbb{C}$$

of $(D_Y^{\mathcal{B}_n})_{\mathbb{C}}$. For a diagram $E : \mathcal{H}_Y \to \mathbb{C}$, the left Kan extension $\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} E$ is obtained as in the non-pointed case by first constructing the left Kan extension $\lambda^E : E \to L_1 \circ D_Y$ and then successively extend it to $\overline{L_1}, L_2, \ldots, L_n, \overline{L_n}$. We then set $\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} E = \overline{L_n}$ with the universal morphism

$$\lambda^E : E \to L_1 \circ D_Y = (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} E) \circ D_Y^{\mathcal{B}_n}.$$

As in the non-pointed case, for each object C in $\mathbb{C}$, there is a unique morphism $e_C : C \to (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_C)(Y)$ such that

$$(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_C)(p^Y_H) \circ e_C = \lambda^{\Delta_C}_H$$

for each $H \in \mathcal{H}_Y$. We consider the pullback

$$\begin{tikzcd} \mathrm{Ap}(C) \arrow[r, "\delta_C"] \arrow[d, "\gamma_C"'] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_C)(X) \arrow[d, "(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_C)(q)"] \\ C \arrow[r, "e_C"'] & (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_C)(Y) \end{tikzcd}$$

and define the endofunctor $\mathrm{Ap}\colon \mathbb{C} \to \mathbb{C}$ and the natural transformation $\gamma\colon \mathrm{Ap} \to 1_{\mathbb{C}}$ as in the non-pointed case.

The pointed versions of Theorems 3.1 and 4.1 become the following ones, whose proofs are omitted since they are completely analogous to the non-pointed case. Remark 3.2 on the existence of colimits needed is still valid for Theorem 6.1.

Theorem 6.1. *Let $\mathbb{C}$ be a finitely bicomplete pointed category and $\alpha \vdash \beta$ an exactness sequent as described in (6). We consider the following statements:*

1. $\alpha \vdash_{\mathbb{C}} \beta$;
2. *for any diagram $E\colon \mathcal{H}_Y \to \mathbb{C}$, the morphism $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} E)(q)$ is a strong epimorphism in $\mathbb{C}$;*
3. *for any object C in $\mathbb{C}$, the morphism $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_C)(q)$ is a strong epimorphism in $\mathbb{C}$;*
4. *for any object C in $\mathbb{C}$, the morphism $\gamma_C\colon \mathrm{Ap}(C) \to C$ is a strong epimorphism in $\mathbb{C}$.*

We always have the implications $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftarrow 4$. Moreover, if $\mathbb{C}$ is regular, we also have the remaining implication $3 \Rightarrow 4$.

Theorem 6.2. *Let $\mathbb{V}$ be a pointed variety of universal algebras. Let also $\alpha \vdash \beta$ be an exactness sequent as described in (6). Then, the following statements are equivalent:*

1. $\alpha \vdash_{\mathbb{V}} \beta$;
2. *for any diagram $E\colon \mathcal{H}_Y \to \mathbb{V}$, the morphism $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} E)(q)$ is surjective;*
3. *for any $\mathbb{V}$-algebra A, the morphism $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_A)(q)$ is surjective;*
4. *for any $\mathbb{V}$-algebra A, the morphism $\gamma_A\colon \mathrm{Ap}(A) \to A$ is surjective;*
5. *the morphism $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\mathrm{Fr}(\{\star\})})(q)$ is surjective;*
6. *the morphism $\gamma_{\mathrm{Fr}(\{\star\})}\colon \mathrm{Ap}(\mathrm{Fr}(\{\star\})) \to \mathrm{Fr}(\{\star\})$ is surjective;*
7. *the element $e_{\mathrm{Fr}(\{\star\})}(\star)$ is in the image of $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\mathrm{Fr}(\{\star\})})(q)$;*
8. *the element $\star$ is in the image of $\gamma_{\mathrm{Fr}(\{\star\})}\colon \mathrm{Ap}(\mathrm{Fr}(\{\star\})) \to \mathrm{Fr}(\{\star\})$.*

6.1 Unitality

A finitely complete pointed category $\mathbb{C}$ is *unital* [5] if for any two objects A and B, the induced morphisms $(1_A, 0)\colon A \to A \times B$ and $(0, 1_B)\colon B \to A \times B$ to the product $A \times B$ of A and B are jointly strongly epimorphic. This property can be stated in the form $\alpha \vdash_{\mathbb{C}} \beta$ for an exactness sequent $\alpha \vdash \beta$ as described in (6). In the display of particular sketches below, we use the same abbreviations as in Section 5. Let $\mathcal{B}_0$ be the sketch described in condition (Ax 0 *) and let $\mathbb{A}_1$ be the discrete category on three objects A, B and Z. Let $\mathcal{A}_1 \supseteq \mathcal{B}_0$ be the underlying sketch of $\mathbb{A}_1$ further equipped with the convergence conditions requiring Z to represent a zero object. Let $\mathcal{B}_1 \supseteq \mathcal{A}_1$ be the sketch displayed as

$$\begin{tikzcd}
& Y \arrow[dl, shift right, "p_A"'] \arrow[dr, shift left, "p_B"] & \\
A \arrow[ur, shift right, "l"'] \arrow[rr, shift left, "0_{A,B}"] \arrow[dr, shift left, "0_{A,Z}"] & & B \arrow[ul, shift left, "r"] \arrow[ll, shift left, "0_{B,A}"] \arrow[dl, shift right, "0_{B,Z}"'] \\
& Z \arrow[ul, shift left, "0_{Z,A}"] \arrow[ur, shift right, "0_{Z,B}"'] &
\end{tikzcd}$$

conditions from $\mathcal{A}_1$ together with:
(Y, p_A, p_B) represents the product of A and B
$0_{A,B} = 0_{Z,B} \cdot 0_{A,Z}$
$0_{B,A} = 0_{Z,A} \cdot 0_{B,Z}$
$p_A \cdot l = 1_A$
$p_B \cdot l = 0_{A,B}$
$p_A \cdot r = 0_{B,A}$
$p_B \cdot r = 1_B$.

Let $\mathbb{A}_2$ be the path category of the graph

$$\begin{tikzcd}
& X \arrow[d, "q"] & \\
& Y \arrow[dl, shift right, "p_A"'] \arrow[dr, shift left, "p_B"] & \\
A \arrow[uur, bend left, "f"] \arrow[ur, shift right, "l"'] \arrow[rr, shift left, "0_{A,B}"] \arrow[dr, shift left, "0_{A,Z}"] & & B \arrow[uul, bend right, "g"'] \arrow[ul, shift left, "r"] \arrow[ll, shift left, "0_{B,A}"] \arrow[dl, shift right, "0_{B,Z}"'] \\
& Z \arrow[ul, shift left, "0_{Z,A}"] \arrow[ur, shift right, "0_{Z,B}"'] &
\end{tikzcd}$$

equipped with the conditions $q \cdot f = l$, $q \cdot g = r$, $0_{AZ} \cdot 0_{ZA} = 1_Z$, $0_{BZ} \cdot 0_{ZB} = 1_Z$ and the last six commutativity conditions written in the above definition of $\mathcal{B}_1$. We set $\mathcal{B}_2 = \mathcal{A}_2 \supseteq \mathcal{B}_1$ to be the underlying sketch of $\mathbb{A}_2$ together with the commutativity and convergence conditions from $\mathcal{B}_1$. Finally we define $\mathcal{A}$ and $\mathcal{B}$ as prescribed by condition (Ax 4) for the arrow $q\colon X \to Y$ in $\mathcal{B}_2$. We have constructed an exactness sequent

$$\begin{tikzcd}
\varnothing \arrow[r, "\beta_0"] \arrow[rrrrrrr, bend right, "\alpha"'] & \mathcal{B}_0 \arrow[r, "\alpha_1"] & \mathcal{A}_1 \arrow[r, "\beta_1"] & \mathcal{B}_1 \arrow[r, "\alpha_2"] & \mathcal{A}_2 \arrow[r, "\beta_2"] & \mathcal{B}_2 \arrow[rr, "\omega"] & & \mathcal{A} \arrow[r, "\beta"] & \mathcal{B}
\end{tikzcd}$$

satisfying conditions (Ax 0 *), (Ax 1)–(Ax 5) and such that, for any finitely complete pointed category $\mathbb{C}$, $\alpha \vdash_{\mathbb{C}} \beta$ holds if and only if $\mathbb{C}$ is unital. The graph $\mathcal{H}_Y$ of condition (Ax 5) can be chosen to be the graph with two objects $*_A$ and $*_B$ and no arrows, and we set $D_Y(*_A) = A$, $D_Y(*_B) = B$, $p^Y_{*_A} = p_A$ and $p^Y_{*_B} = p_B$.

Given a finitely bicomplete pointed category $\mathbb{C}$ and an object $C \in \mathbb{C}$, the Kan extension $\mathrm{Lan}_{D_Y^{\mathcal{B}_2}} \Delta_C$ is constructed as follows. Using the pointwise pointed Kan extension formula, we know that $(\mathrm{Lan}_{D_Y} \Delta_C)(A) = C = (\mathrm{Lan}_{D_Y} \Delta_C)(B)$ and $(\mathrm{Lan}_{D_Y} \Delta_C)(Z) = 0$. The extension $\mathrm{Lan}_{\beta_1 \circ D_Y} \Delta_C$ of $\mathrm{Lan}_{D_Y} \Delta_C$ as a $\mathcal{B}_1$-structure is described by:

$$\begin{tikzcd}
& C \times C \arrow[dl, shift right, "p_1"'] \arrow[dr, shift left, "p_2"] & \\
C \arrow[ur, shift right, "(1_C,0)"'] \arrow[rr, shift left, "0"] \arrow[dr, shift right, "0"'] & & C \arrow[ul, shift left, "(0,1_C)"] \arrow[ll, shift left, "0"] \arrow[dl, shift left, "0"] \\
& 0 \arrow[ul, shift right, "0"'] \arrow[ur, shift left, "0"] &
\end{tikzcd}$$

Finally, we can compute that $\mathrm{Lan}_{\alpha_2 \circ \beta_1 \circ D_Y} \Delta_C = \mathrm{Lan}_{D_Y^{\mathcal{B}_2}} \Delta_C$ is described by:

$$\begin{tikzcd}
& C + C \arrow[d, "\left(\begin{smallmatrix} 1_C & 0 \\ 0 & 1_C \end{smallmatrix}\right)"] & \\
& C \times C \arrow[dl, shift right, "p_1"'] \arrow[dr, shift left, "p_2"] & \\
C \arrow[uur, bend left, "\iota_1"] \arrow[ur, shift right, "(1_C,0)"'] \arrow[rr, shift left, "0"] \arrow[dr, shift right, "0"'] & & C \arrow[uul, bend right, "\iota_2"'] \arrow[ul, shift left, "(0,1_C)"] \arrow[ll, shift left, "0"] \arrow[dl, shift left, "0"] \\
& 0 \arrow[ul, shift right, "0"'] \arrow[ur, shift left, "0"] &
\end{tikzcd}$$

Applying Theorem 6.1 (together with Remark 3.2) to this situation gives us the well-known fact that a finitely complete pointed category $\mathbb{C}$ with binary copowers is unital if and only if, for any object $C \in \mathbb{C}$, the morphism $\left(\begin{smallmatrix} 1_C & 0 \\ 0 & 1_C \end{smallmatrix}\right) : C+C \to C \times C$ is a strong epimorphism. Moreover, it also gives that a regular pointed category $\mathbb{C}$ with binary copowers is unital if and only if, for any object $C \in \mathbb{C}$, the projection γ_C in the pullback

$$\begin{tikzcd}
\mathrm{Ap}(C) \arrow[r, "\delta_C"] \arrow[d, "\gamma_C"'] \arrow[dr, phantom, very near start, "\lrcorner"] & C + C \arrow[d, "\left(\begin{smallmatrix} 1_C & 0 \\ 0 & 1_C \end{smallmatrix}\right)"] \\
C \arrow[r, "e_C = (1_C, 1_C)"'] & C \times C
\end{tikzcd}$$

is a regular epimorphism; a result which already appears in [30]. Theorem 6.2 tells us that a pointed variety $\mathbb{V}$ is unital if and only if the element (x, x) is in the image of the morphism $h \colon \mathrm{Fr}(\{x_1, x_2\}) \to \mathrm{Fr}(\{x\})^2$ determined by $h(x_1) = (x, 0)$ and $h(x_2) = (0, x)$ where 0 is the unique constant in the theory of $\mathbb{V}$. This happens if and only if there exists a binary term $u(x_1, x_2)$ in the theory of $\mathbb{V}$ satisfying the identities $u(x, 0) = x$ and $u(0, x) = x$. This characterization goes back at least to [3].

6.2 Strong unitality

Strongly unital categories have been introduced in [5]. In [3], they have been characterized as finitely complete pointed categories such that, for any diagram

$$\begin{tikzcd} A \arrow[r, shift left, tail, "s"] & X \arrow[l, shift left, two heads, "f"] \arrow[r, shift right, two heads, "g"'] & B \arrow[l, shift right, tail, "t"'] \end{tikzcd}$$

where $f \circ s = 1_A$, $g \circ t = 1_B$ and $f \circ t = 0$, the induced morphism $(f, g)\colon X \to A \times B$ is a strong epimorphism. We can express this property in the form $\alpha \vdash_{\mathbb{C}} \beta$ for an exactness sequent $\alpha \vdash \beta$ as described in (6). Let $\mathcal{B}_0$ be as in condition (Ax 0 *) and let $\mathcal{G}_1$ be the graph

$$\begin{tikzcd} Z & A \arrow[r, shift left, "s"] & X \arrow[l, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[l, shift right, "t"'] \end{tikzcd}$$

to which we add, for any pair of object (V, W), an arrow $0_{V,W}\colon V \to W$. Let $\mathbb{A}_1$ be the path category of $\mathcal{G}_1$ equipped with the conditions $f \cdot s = 1_A$, $g \cdot t = 1_B$, $f \cdot t = 0_{B,A}$, $0_{Z,Z} = 1_Z$ and all conditions of the form $0_{V_2,W} \cdot h = 0_{V_1,W}$ or of the form $h \cdot 0_{W,V_1} = 0_{W,V_2}$ for an arrow $h\colon V_1 \to V_2$ and objects V_1, V_2, W in $\mathcal{G}_1$. Notice that $\mathbb{A}_1$ is a finite pointed category with zero object Z (although we do not need in general that Z is a zero object of $\mathbb{A}_1$). Let $\mathcal{A}_1 \supseteq \mathcal{B}_0$ be the underlying sketch of $\mathbb{A}_1$ further equipped with the convergence conditions from $\mathcal{B}_0$. Let $\mathcal{B}_1 \supseteq \mathcal{A}_1$ be the sketch displayed below (using the same abbreviations as in Section 5 and also omitting the zero morphisms from $\mathcal{A}_1$).

$$\begin{tikzcd} & & Y \arrow[dl, "p_A"'] \arrow[dr, "p_B"] & \\ Z & A \arrow[r, shift left, "s"] & X \arrow[u, "q"] \arrow[l, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[l, shift right, "t"'] \end{tikzcd}$$

conditions from $\mathcal{A}_1$ together with:
(Y, p_A, p_B) represents the product of A and B
$p_A \cdot q = f$
$p_B \cdot q = g$.

Let $\mathcal{A}$ and $\mathcal{B}$ be the sketches given by condition (Ax 4) for the arrow $q\colon X \to Y$ in $\mathcal{B}_1$. We have constructed an exactness sequent

$$\begin{tikzcd} \varnothing \arrow[r, "\beta_0"] \arrow[rrrr, bend right, "\alpha"'] & \mathcal{B}_0 \arrow[r, "\alpha_1"] & \mathcal{A}_1 \arrow[r, "\beta_1"] & \mathcal{B}_1 \arrow[r, "\omega"] & \mathcal{A} \arrow[r, "\beta"] & \mathcal{B} \end{tikzcd}$$

satisfying conditions (Ax 0 *), (Ax 1)–(Ax 5) and such that, for any finitely complete pointed category $\mathbb{C}$, $\alpha \vdash_{\mathbb{C}} \beta$ holds if and only if $\mathbb{C}$ is strongly unital. As before, the graph $\mathcal{H}_Y$ is chosen to be the graph with two objects $*_A$ and $*_B$ and no arrows, and we set $D_Y(*_A) = A$, $D_Y(*_B) = B$, $p^Y_{*_A} = p_A$ and $p^Y_{*_B} = p_B$.

For an object C in a finitely bicomplete pointed category $\mathbb{C}$, the left Kan extension $\mathrm{Lan}_{D_Y} \Delta_C$ can be computed via the pointwise pointed left Kan extension formula, and can be extended to the Kan extension $\mathrm{Lan}_{D_Y^{\mathcal{B}_1}} \Delta_C$ displayed below

$$\begin{tikzcd} & & C \times 2C \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] & \\ 0 & C \arrow[r, shift left, "\iota_1"] & 3C \arrow[u, "h"] \arrow[l, shift left, "\begin{pmatrix} 1_C \\ 0 \\ 0 \end{pmatrix}"] \arrow[r, shift right, "\begin{pmatrix} \iota_2' \\ \iota_2' \\ \iota_1' \end{pmatrix}"'] & 2C \arrow[l, shift right, "\begin{pmatrix} \iota_3 \\ \iota_2 \end{pmatrix}"'] \end{tikzcd}$$

where $h = \begin{pmatrix} 1_C & \iota_2' \\ 0 & \iota_2' \\ 0 & \iota_1' \end{pmatrix}$. Theorem 6.1 tells us in this case that a finitely complete pointed category $\mathbb{C}$ with finite copowers is strongly unital if and only if, for any object $C \in \mathbb{C}$, the morphism $\begin{pmatrix} 1_C & \iota_2' \\ 0 & \iota_2' \\ 0 & \iota_1' \end{pmatrix} : 3C \to C \times 2C$ is a strong epimorphism. Moreover, it also says that a regular pointed category $\mathbb{C}$ with finite copowers is strongly unital if and only if, for each object $C \in \mathbb{C}$, the projection γ_C in the pullback

$$\begin{tikzcd} \mathrm{Ap}(C) \arrow[r, "\delta_C"] \arrow[d, "\gamma_C"'] \arrow[dr, phantom, "\lrcorner", very near start] & 3C \arrow[d, "\begin{pmatrix} 1_C & \iota_2' \\ 0 & \iota_2' \\ 0 & \iota_1' \end{pmatrix}"] \\ C \arrow[r, "e_C=(1_C,\iota_1')"'] & C \times 2C \end{tikzcd}$$

is a regular epimorphism; a result which already appears in [30]. Theorem 6.2 applied here says that a pointed variety $\mathbb{V}$ is strongly unital if and only if the element (x, x_1) is in the image of the morphism $h\colon \mathrm{Fr}(\{y_1, y_2, y_3\}) \to \mathrm{Fr}(\{x\}) \times \mathrm{Fr}(\{x_1, x_2\})$ as defined above, i.e., if and only if there exists a ternary term $p(y_1, y_2, y_3)$ in the theory of $\mathbb{V}$ satisfying the identities $p(x, 0, 0) = x$ and $p(x_2, x_2, x_1) = x_1$ (where 0 is the unique constant in the theory of $\mathbb{V}$). This characterization goes back at least to [3].

6.3 Additional examples

Matrix conditions have been introduced in [28] and generalized in [29]. To each extended matrix $(M, \mathcal{X})$ of terms in the theory $\mathrm{Th}[\mathbf{Set}_*]$ of pointed sets (in the sense of [29]) is associated the exactness property on a regular pointed category to have $(M, \mathcal{X})$-closed relations. The exactness properties of being a unital category or a strongly unital category are examples of such properties, but also the property of being a subtractive category [27]. For each such extended matrix $(M, \mathcal{X})$, one can construct an exactness sequent $\alpha \vdash \beta$ as described in (6) with $n = 1$ such that a regular pointed category $\mathbb{C}$ has $(M, \mathcal{X})$-closed relations if and only if $\alpha \vdash_{\mathbb{C}} \beta$ holds. Theorem 6.1 gives in this particular case the approximate co-operation characterization of these properties, initiated in [8] and generalized in [30, 19, 22]. Theorem 6.2 gives the characterization of pointed varieties satisfying those properties proved in [28, 29]. The non-pointed version of this also holds for extended matrices $(M, \mathcal{X})$ of terms in the theory $\mathrm{Th}[\mathbf{Set}]$ of sets as mentioned in Subsection 5.3 and explained in [24]. This includes the following exactness properties in the regular context: being a Mal'tsev category [11], being an n-permutable category [10] and being a majority category [17].

The property on a pointed regular category $\mathbb{C}$ to be a *normal category* [31] (i.e., every regular epimorphism is a normal epimorphism) can be expressed in the form $\alpha \vdash_{\mathbb{C}} \beta$ for an exactness sequent $\alpha \vdash \beta$ as described in (6). To see this, one can reformulate the property as follows: given any morphism f, considering its kernel pair (R, r_1, r_2) and its kernel (K, k), then any morphism g such that $g \circ k = 0$ satisfies $g \circ r_1 = g \circ r_2$, or in other words, the equalizer of $g \circ r_1$ and $g \circ r_2$ is a strong epimorphism. Theorem 6.1 applied to this particular case says that a regular finitely cocomplete pointed category $\mathbb{C}$ is normal if and only if the codiagonal $\nabla_C\colon 2C \to C$ is a normal epimorphism for all objects C in $\mathbb{C}$. Theorem 6.2 tells us that a pointed variety is normal (also called a *variety with ideals* according to [14]) if and only if the variables x_1 and x_2 are coequalized by the cokernel of the kernel of the codiagonal $\mathrm{Fr}(\{x_1, x_2\}) \to \mathrm{Fr}(\{x\})$. Making explicit the equality in this cokernel, one gets the characterization of varieties with ideals from [14].

The property on a finitely complete pointed category to have normal projections [26] and the property on a regular pointed category with coequalizers to have products commuting with coequalizers can be treated similarly to normality. For the latter, one should use the second part of point 3 in Proposition 2.9 in [18]. In both cases, Theorem 6.2 gives us the free algebras one should consider to extract the algebraic conditions characterizing these properties for pointed varieties, appearing in [26] and [18] respectively. The remaining task to get these characterizations is to express the equality of some elements in a certain coequalizer.

7 Further remarks

Remark 7.1. It has recently been shown (see [19, 20, 21, 22, 23, 24]) that the study of exactness properties is strongly related with the study of essentially algebraic categories [1] (i.e., locally presentable categories [15]). In a similar way as in [24], Theorem 4.1 can be generalized to any essentially algebraic category. We treated only the varietal case in the body of the paper for the sake of simplicity. Using the notation from [24], we now fix an essentially algebraic theory $\Gamma = (S, \Sigma, E, \Sigma_t, \mathrm{Def})$. We recall that the category of models $\mathrm{Mod}(\Gamma)$ has a (strong epi, mono)-factorization system (but strong epimorphisms are in general not sort-wise surjective). For each sort $s \in S$, we denote by $\{\star_s\}$ the S-sorted set with a single element $\star_s$ in the sort s and nothing in the other sorts. The homomorphism $e_{\mathrm{Fr}(\{\star_s\})} \colon \mathrm{Fr}(\{\star_s\}) \to (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\mathrm{Fr}(\{\star_s\})})(Y)$ from the free Γ-model on $\{\star_s\}$ induces an element $(e_{\mathrm{Fr}(\{\star_s\})})_s(\star_s)$ in $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\mathrm{Fr}(\{\star_s\})})(Y)_s$. We can then generalize Theorem 4.1 as follows.

Theorem 7.2. *Let $\Gamma = (S, \Sigma, E, \Sigma_t, \mathrm{Def})$ be an essentially algebraic theory. Let also $\alpha \vdash \beta$ be an exactness sequent as described in (3). The following statements are equivalent:*

1. $\alpha \vdash_{\mathrm{Mod}(\Gamma)} \beta$*;*
2. *for any diagram $E \colon \mathcal{H}_Y \to \mathrm{Mod}(\Gamma)$, the morphism $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} E)(q)$ is a strong epimorphism;*
3. *for any Γ-model A, the morphism $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_A)(q)$ is a strong epimorphism;*
4. *for each sort $s \in S$, the morphism $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\mathrm{Fr}(\{\star_s\})})(q)$ is a strong epimorphism;*
5. *for each sort $s \in S$, the element $(e_{\mathrm{Fr}(\{\star_s\})})_s(\star_s)$ is in the image of $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\mathrm{Fr}(\{\star_s\})})(q)$.*

Proof. The equivalences $1 \Leftrightarrow 2 \Leftrightarrow 3$ follow immediately from Theorem 3.1. The implications $3 \Rightarrow 4 \Rightarrow 5$ being trivial, it remains to prove $5 \Rightarrow 1$. So let $\overline{F_n}$ be any $\mathcal{B}_n$-structure in $\mathrm{Mod}(\Gamma)$ and let us show that $\overline{F_n}(q)$ is a strong epimorphism. Given any sort $s \in S$ and any element $y \in \overline{F_n}(Y)_s$, we must show that y is in the image of $\overline{F_n}(q)$. We consider the unique homomorphism of Γ-models $f \colon \mathrm{Fr}(\{\star_s\}) \to \overline{F_n}(Y)$ such that $f_s(\star_s) = y$. Considering the morphism of $\mathcal{B}_n$-structures $\nu \colon \mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\overline{F_n}(Y)} \to \overline{F_n}$ as in the proof of Theorem 3.1, we know that the following diagram commutes.

$$\begin{tikzcd}
(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\mathrm{Fr}(\{\star_s\})})(X) \arrow[r, "(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_f)_X"] \arrow[d, "(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\mathrm{Fr}(\{\star_s\})})(q)"'] & (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\overline{F_n}(Y)})(X) \arrow[r, "\nu_X"] \arrow[d, "(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\overline{F_n}(Y)})(q)"'] & \overline{F_n}(X) \arrow[d, "\overline{F_n}(q)"] \\
(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\mathrm{Fr}(\{\star_s\})})(Y) \arrow[r, "(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_f)_Y"'] & (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\overline{F_n}(Y)})(Y) \arrow[r, two heads, "\nu_Y"] & \overline{F_n}(Y) \\
\mathrm{Fr}(\{\star_s\}) \arrow[u, "e_{\mathrm{Fr}(\{\star_s\})}"] \arrow[r, "f"'] & \overline{F_n}(Y) \arrow[u, tail, "e_{\overline{F_n}(Y)}"'] \arrow[ur, bend right, "1_{\overline{F_n}(Y)}"'] &
\end{tikzcd}$$

Since $y = f_s(\star_s) = (\nu_Y \circ (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_f)_Y \circ e_{\mathrm{Fr}(\{\star_s\})})_s(\star_s)$ and since $(e_{\mathrm{Fr}(\{\star_s\})})_s(\star_s)$ is in the image of $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\mathrm{Fr}(\{\star_s\})})(q)$, we know that y is in the image of $\overline{F_n}(q) \circ \nu_X \circ (\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_f)_X$ and thus in the image of $\overline{F_n}(q)$. $\square$

For the pointed version of this theorem, we recall (see e.g. [19]) that an essentially algebraic theory $\Gamma = (S, \Sigma, E, \Sigma_t, \mathrm{Def})$ gives rise to a pointed category $\mathrm{Mod}(\Gamma)$ if and only if, there exists in Γ, for each sort $s \in S$, a unique everywhere-defined constant term 0^s of sort s, and every constant term is everywhere-defined (and thus equal to a 0^s). The pointed version of Theorem 7.2, proved in an analogous way, is the following theorem which generalizes Theorem 6.2.

Theorem 7.3. *Let* $\Gamma = (S, \Sigma, E, \Sigma_t, \mathrm{Def})$ *be an essentially algebraic theory whose category of models* $\mathrm{Mod}(\Gamma)$ *is pointed. Let also* $\alpha \vdash \beta$ *be an exactness sequent as described in (6). The following statements are equivalent:*

1. $\alpha \vdash_{\mathrm{Mod}(\Gamma)} \beta$;
2. *for any diagram* $E \colon \mathcal{H}_Y \to \mathrm{Mod}(\Gamma)$, *the morphism* $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} E)(q)$ *is a strong epimorphism;*
3. *for any* Γ*-model* A, *the morphism* $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_A)(q)$ *is a strong epimorphism;*
4. *for each sort* $s \in S$, *the morphism* $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\mathrm{Fr}(\{\star_s\})})(q)$ *is a strong epimorphism;*
5. *for each sort* $s \in S$, *the element* $(e_{\mathrm{Fr}(\{\star_s\})})_s(\star_s)$ *is in the image of* $(\mathrm{Lan}_{D_Y^{\mathcal{B}_n}} \Delta_{\mathrm{Fr}(\{\star_s\})})(q)$.

Remark 7.4. Notice that in an exactness sequent $\alpha \vdash \beta$

$$\begin{tikzcd}
\varnothing \arrow[r, "\alpha_1"] \arrow[rrrrrrr, bend right, "\alpha"'] & \mathcal{A}_1 \arrow[r, "\beta_1"] & \mathcal{B}_1 \arrow[r, "\alpha_2"] & \cdots \arrow[r, "\alpha_n"] & \mathcal{A}_n \arrow[r, "\beta_n"] & \mathcal{B}_n \arrow[rr, tail, "\omega"] & & \mathcal{A} \arrow[r, "\beta"] & \mathcal{B}
\end{tikzcd}$$

as described in (3), the subsketch inclusion α is in general not unconditional of finite kind. One can construct from $q \in \mathcal{B}_n$ a constructible subsketch inclusion $\beta' \colon \mathcal{B}_n \to \mathcal{B}'$ such that the property

> 'for any $\mathcal{A}_1$-structure F_1 in $\mathbb{C}$, considering its extension $\overline{F_1}$ as a $\mathcal{B}_1$-structure, for any extension of it as an $\mathcal{A}_2$-structure F_2, considering its extension $\overline{F_2}$ as a $\mathcal{B}_2$-structure, ..., for any extension of $\overline{F_{n-1}}$ as an $\mathcal{A}_n$-structure F_n, its extension $\overline{F_n}$ as a $\mathcal{B}_n$-structure extends to a $\mathcal{B}'$-structure'

represented by the sequence

$$\varnothing \xrightarrow{\alpha_1} \mathcal{A}_1 \xrightarrow{\beta_1} \mathcal{B}_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} \mathcal{A}_n \xrightarrow{\beta_n} \mathcal{B}_n \xrightarrow{\beta'} \mathcal{B}'$$

holds for a regular finitely cocomplete category $\mathbb{C}$ if and only if $\alpha \vdash_{\mathbb{C}} \beta$ holds. This new exactness property could be seen as an 'n-th order exactness property' (although this notion has not yet been formalized) for which the odd inclusions $\alpha_1, \dots, \alpha_n$ are unconditional of finite kind and the even inclusions $\beta_1, \dots, \beta_{n-1}, \beta' \circ \beta_n$ are constructible. We build $\beta' \colon \mathcal{B}_n \to \mathcal{B}'$ from q via the following two steps:

1. include an object R (not already in $\mathcal{B}_n$) and arrows $r_1, r_2 \colon R \rightrightarrows X$, together with the convergence condition '(R, r_1, r_2) represents the kernel pair of q',
2. then include the convergence condition '(Y, q) represents the coequalizer of r_1 and r_2'.

The stated equivalence between the two mentioned exactness properties follows immediately from the fact that in a regular category, strong epimorphisms coincide with regular epimorphisms.

Conjecture 7.5. Theorem 4.1 gives, for an exactness sequent $\alpha \vdash \beta$ as in (3), a characterization of the condition $\alpha \vdash_{\mathbb{V}} \beta$ on a variety $\mathbb{V}$. As explained in Section 5 for particular examples, it seems that the statement 8 of that theorem can always be reformulated as a Mal'tsev condition. However, we have not been able to formally prove this fact in general but we conjecture it is always the case. By a Mal'tsev condition, we mean here a disjunction of (non-necessarily nested) strong Mal'tsev conditions (where 'strong Mal'tsev condition' is understood in its classical sense, see e.g. [33]).

Conjecture 7.6. We also conjecture that for any exactness sequent $\alpha \vdash \beta$ as described in (3), there exists another such exactness sequent $\alpha' \vdash \beta'$ for which the procedures (b), (f) and (g) from condition (Ax 3) are not used and for which $\alpha' \vdash_{\mathbb{C}} \beta'$ holds for a finitely bicomplete category $\mathbb{C}$ if and only if $\alpha \vdash_{\mathbb{C}} \beta$ holds.

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